

A unified Erdős-Pósa theorem for constrained cycles

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Abstract

A *doubly group-labeled graph* is an oriented graph with its edges labeled by elements of the direct sum of two groups Γ_1, Γ_2 . A cycle in a doubly group-labeled graph is (Γ_1, Γ_2) -*non-zero* if it is non-zero in both coordinates. Our main result is a generalization of the Flat Wall Theorem of Robertson and Seymour to doubly group-labeled graphs. As an application, we determine all canonical obstructions to the Erdős-Pósa property for (Γ_1, Γ_2) -non-zero cycles in doubly group-labeled graphs. The obstructions imply that the half-integral Erdős-Pósa property always holds for (Γ_1, Γ_2) -non-zero cycles.

Moreover, our approach gives a unified framework for proving packing results for constrained cycles in graphs. For example, as immediate corollaries we recover the Erdős-Pósa property for cycles and S -cycles and the half-integral Erdős-Pósa property for odd cycles and odd S -cycles. Furthermore, we recover Reed's Escher-wall Theorem.

We also prove many new packing results as immediate corollaries. For example, we show that the half-integral Erdős-Pósa property holds for cycles not homologous to zero, odd cycles not homologous to zero, and S -cycles not homologous to zero. Moreover, the (full) Erdős-Pósa property holds for S_1 - S_2 -cycles and cycles not homologous to zero on an orientable surface. Finally, we also describe the canonical obstructions to the Erdős-Pósa property for cycles not homologous to zero and for odd S -cycles.

1 Introduction

Erdős and Pósa proved in a seminal paper [5] that for every positive integer k , there exists a constant $f(k)$ such that every graph G either contains k pairwise disjoint¹ cycles, or a set $X \subseteq V(G)$ of size at most $f(k)$ such that $G - X$ has no cycle. This result has had numerous extensions and generalizations to cycles satisfying further constraints. Specific examples of families of cycles which have been studied are: cycles of odd length [16], cycles of length at least ℓ for some $\ell \in \mathbb{N}$ [1, 7], disjoint S -cycles where each cycle intersects a prescribed set of vertices S [11, 15], S -cycles of odd length [12], and S -cycles of length at least ℓ again for some fixed value ℓ [2]. In each case, the goal is to show the *Erdős-Pósa property* for a given family of cycles: that is, the existence of a function f such

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¹In this paper disjoint always means vertex-disjoint.

that for all k , every graph either has k disjoint cycles satisfying the desired constraints, or a set X of at most $f(k)$ vertices intersecting every such cycle.

In this paper, we consider the analogous problem in group-labeled graphs. This approach provides a unified framework for proving Erdős-Pósa results for families of cycles with additional constraints on the cycles. In particular, we obtain all of the above results as corollaries of our main theorems.

In this article, the graphs we consider may have loops and parallel edges. An *oriented graph* is a graph \vec{G} along with two functions $\text{head}_{\vec{G}}$ and $\text{tail}_{\vec{G}}$ from $E(G)$ to $V(G)$ such that for every non-loop edge e with distinct endpoints u and v , we have $\text{head}_{\vec{G}}(e) \in \{u, v\}$, and $\text{tail}_{\vec{G}}(e) = \{u, v\} \setminus \text{head}_{\vec{G}}(e)$. In the case when e is a loop on a vertex v , we have $\{\text{head}_{\vec{G}}(e), \text{tail}_{\vec{G}}(e)\} = \{u, v\}$. Given an oriented graph \vec{G} , the undirected graph G with the same vertex and edge set as \vec{G} is the graph obtained by *forgetting* the orientations. Notationally, we will always indicate oriented graphs with an arrow and undirected graphs without. When there can be no confusion, we will simply use *tail* and *head* in the place of $\text{tail}_{\vec{G}}$ and $\text{head}_{\vec{G}}$.

Let $\Gamma, \Gamma_1, \Gamma_2$ be (possibly infinite, possibly non-abelian) groups. Even though our theorems hold for non-abelian groups, we will always use $+$ to denote the group operation and 0 for the identity element. This follows the convention in [3], and is mainly because the adjective *non-zero* is so ubiquitous in the literature. A Γ -labeled graph is a pair (\vec{G}, γ) , where \vec{G} is an oriented graph and $\gamma : E(\vec{G}) \rightarrow \Gamma$. Group-labeled graphs have been extensively studied; see [3, 10, 21, 22]. Let v be an end of an edge e . We define $\gamma(e, v) := \gamma(e)$ if $v = \text{head}(e)$ and $\gamma(e, v) = -\gamma(e)$ if $v = \text{tail}(e)$. If \vec{H} is a subgraph of \vec{G} , we abuse notation by letting γ also denote the restriction of γ to $E(\vec{H})$.

A *walk* in (\vec{G}, γ) is a walk in G . Let $W = v_0 e_1 v_1 e_2 v_2 \dots e_\ell v_\ell$ be a walk in (\vec{G}, γ) . The *length* of W is ℓ , and its *ends* are v_0 and v_ℓ . We say that W is a *path* if v_0, \dots, v_ℓ are distinct and is a *cycle* if $v_0 = v_\ell$ and v_1, \dots, v_ℓ are distinct. The *group-value* of W , denoted $\gamma(W)$, is defined to be $\gamma(e_1, v_1) + \dots + \gamma(e_\ell, v_\ell)$. We say W is Γ -zero if $\gamma(W) = 0$. Otherwise, we say that W is Γ -non-zero. Note that if W is a cycle, it is naturally equipped with a starting point and an orientation (since it is a walk). We remark that if C_1 and C_2 are cycles such that $E(C_1) = E(C_2)$, then C_1 is Γ -zero if and only if C_2 is Γ -zero (see Section 3 for a proof of this claim). Thus, whether a cycle C is Γ -zero or Γ -non-zero only depends on its edge set. This is easy to see if Γ is abelian, since in this case $\gamma(C_1) \in \{\gamma(C_2), -\gamma(C_2)\}$.

We consider a stronger notion of non-zero cycles in the case that (\vec{G}, γ) is a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph². For $i \in [2]$, we let γ_i be the projection of γ onto Γ_i . A walk W is Γ_i -zero if $\gamma_i(W) = 0$. Otherwise, W is Γ_i -non-zero. If W is Γ_1 -non-zero and Γ_2 -non-zero, then we say W is (Γ_1, Γ_2) -non-zero.

Let \mathcal{G} be a class of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. We say that \mathcal{G} has the *Erdős-Pósa-property for (Γ_1, Γ_2) -non-zero cycles* if there exists a function $f_{\mathcal{G}}(k)$ such that every $(\vec{G}, \gamma) \in \mathcal{G}$ either contains k pairwise disjoint (Γ_1, Γ_2) -non-zero cycles, or a set $X \subseteq V(G)$ of size at most $f_{\mathcal{G}}(k)$ such that $(\vec{G} - X, \gamma)$ has no (Γ_1, Γ_2) -non-

²We denote by $\Gamma_1 \oplus \Gamma_2$ the direct sum of Γ_1 and Γ_2 . The elements of $\Gamma_1 \oplus \Gamma_2$ can be written as (α_1, α_2) with $\alpha_i \in \Gamma_i$ and the group operation is induced by the coordinate-wise group operation of Γ_1 and Γ_2 , respectively.

zero cycle. We say that $f_{\mathcal{G}}(k)$ is an *Erdős-Pósa function* for (Γ_1, Γ_2) -non-zero cycles in \mathcal{G} .

In Section 4, we show that the Erdős-Pósa-property for (Γ_1, Γ_2) -non-zero cycles does *not* hold for the class of all $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. However, the following weakening of the Erdős-Pósa-property does hold. We say that a class \mathcal{G} of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs has the *half-integral Erdős-Pósa-property* for (Γ_1, Γ_2) -non-zero cycles if there exists a function $f_{\mathcal{G}}(k)$ such that for every $(\vec{G}, \gamma) \in \mathcal{G}$ either

- (i) there is a collection \mathcal{C} of k (Γ_1, Γ_2) -non-zero cycles such that each vertex of (\vec{G}, γ) is contained in at most two members of \mathcal{C} , or
- (ii) there is a set $X \subseteq V(G)$ of size at most $f_{\mathcal{G}}(k)$ such that $(\vec{G} - X, \gamma)$ has no (Γ_1, Γ_2) -non-zero cycle.

We say that $f_{\mathcal{G}}(k)$ is a *half-integral Erdős-Pósa function* for (Γ_1, Γ_2) -non-zero cycles in \mathcal{G} .

Theorem 1. *For every integer k , there exists an integer $f(k)$ with the following property. Let Γ_1 and Γ_2 be groups and let (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Then, (\vec{G}, γ) contains k (Γ_1, Γ_2) -non-zero cycles such that each vertex of (\vec{G}, γ) is in at most two of these cycles, or there exists a set X of at most $f(k)$ vertices of G such that $(\vec{G} - X, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle.*

We also obtain the (full) Erdős-Pósa-property for (Γ_1, Γ_2) -non-zero cycles for the following restricted class of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. For $i \in [2]$, let Z_i be the set of edges of a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph \vec{G} that lie in a Γ_i -zero cycle. We say (\vec{G}, γ) is *robust* if the following holds for all $i \in [2]$: for all Γ_i -non-zero cycles C_1 and C_2 such that

- $E(C_1) \neq E(C_2)$
- $\emptyset \neq E(C_1) \cap E(C_2) \subseteq Z_i$, and
- C_1 and C_2 have the same starting vertex,

we have $\gamma_i(C_1) \neq \gamma_i(C_2)$. Note that this condition is easier to check if $(\Gamma_1 \oplus \Gamma_2)$ is abelian because $\gamma(C_i)$ is independent of the starting vertex.

Theorem 2. *For every integer k , there exists an integer $f(k)$ with the following property. Let Γ_1 and Γ_2 be groups and let (\vec{G}, γ) be a robust $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Then, (\vec{G}, γ) contains k disjoint (Γ_1, Γ_2) -non-zero cycles or there exists a set of at most $f(k)$ vertices of (\vec{G}, γ) such that $(\vec{G} - X, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle.*

Note that in Theorem 1 and 2, the Erdős-Pósa functions do not depend on the groups Γ_1 and Γ_2 . Also, if either Γ_1 or Γ_2 is the trivial group, then there are no (Γ_1, Γ_2) -non-zero cycles, so the Erdős-Pósa property holds trivially. In addition, if we set $\Gamma_1 = \Gamma_2$ and all edges have the same label in both coordinates, then (Γ_1, Γ_2) -non-zero cycles coincide with Γ -non-zero cycles.

We prove Theorem 1 and 2 via a structure theorem for $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs, which is a refinement of the Flat Wall Theorem of Robertson and

Seymour [18, Theorem 9.8]. Our structure theorem (Theorem 22) is likely of independent interest. For example, Theorem 22 provides canonical obstructions to the Erdős-Pósa property for general constrained cycles that are analogous to the *Escher-walls* described by Reed [16]. These obstructions are defined in Section 4.

Our proof of Theorem 22 extends the techniques developed in [8] and [9]. We introduce a new notion of Γ -odd clique minors for Γ -labeled graphs, which is useful for attacking problems for non-zero cycles. Our definition agrees with the usual notion of odd clique minors when $\Gamma = \mathbb{Z}/2\mathbb{Z}$, but is weaker than the notion of a Γ -labeled clique used in [8]. Our notion is well-defined even in the case that Γ is infinite (note that it will be necessary to consider infinite Γ for some of our applications).

The original draft of Theorem 1 and 2 stated the results under the assumption that the group Γ is abelian; an early manuscript of [14] led us to the observation that both theorems (and their respective proofs) hold without the assumption that Γ is abelian.

We suspect that Theorem 22 will have further applications outside those discussed in this paper. However, as it is rather technical, we defer the statement of Theorem 22 until Section 6 and the proof until Section 9.

We instead discuss some applications of Theorem 1 and 2 in the next section. The rest of the paper is organized as follows. In Section 3, we discuss some preliminaries. In Section 4, we provide a canonical set of obstructions to the Erdős-Pósa for (Γ_1, Γ_2) -non-zero cycles. In Section 5, we introduce our notion of Γ -odd clique minors, and prove a structure theorem for Γ -labeled graphs without a Γ -odd clique minor. In Section 6, we state our version of the Flat Wall Theorem for (Γ_1, Γ_2) -group-labeled graphs (Theorem 22). In Section 7 and 8, we prove some lemmas to be used in the proof of Theorem 22. We prove Theorem 22 in Section 9. We then easily derive Theorem 1 and 2 from Theorem 22 in Section 10 and finish with some further applications.

2 Applications

In this section we illustrate the numerous corollaries of Theorem 1 and 2.

Cycles. Let G be a graph and let \vec{G} be an arbitrary orientation of G . Let e_1, \dots, e_m be an enumeration of $E(\vec{G})$ and define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ by $g(e_i) = (2^i, 2^i)$. Note that (\vec{G}, γ) does not contain any Γ_i -zero cycles, and therefore (\vec{G}, γ) is clearly robust. Since every cycle in G corresponds to a (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) , Theorem 2 implies the original theorem of Erdős and Pósa.

Odd Cycles. Let G be a graph. An *odd cycle* of G is a cycle with an odd number of vertices. Thomassen [19] proved that the Erdős-Pósa-property does not hold for odd cycles. On the other hand, Reed [16] showed that the set of odd cycles has the half-integral Erdős-Pósa property.

Theorem 3 (Reed [16]). *There exists a function f such that for every $k \geq 1$ and graph G , either G contains k odd cycles C_1, \dots, C_k such that every vertex is contained in at most two distinct C_i , or alternatively, there exists a set $X \subseteq V(G)$ such that $G - X$ is bipartite and $|X| \leq f(k)$.*

We derive Theorem 3 as a corollary of Theorem 1 as follows. Let \vec{G} be an arbitrary orientation of G . Define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ by $\gamma(e) = (1, 1)$ for all $e \in E(\vec{G})$. Finish by observing that every odd cycle of G corresponds to a (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) .

S-cycles. Given a graph G and a fixed subset $S \subseteq V(G)$, an S -cycle is a cycle in G containing at least one vertex of S . Recent work of Kakimura, Kawarabayashi and Marx [11] and Pontecorvi and Wollan [15] shows that the Erdős-Pósa property holds for the family of S -cycles.

Theorem 4 ([11, 15]). *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph G , every $S \subseteq V(G)$, and every positive integer k , either G has k disjoint S -cycles or there exists a set of at most $f(k)$ vertices intersecting every S -cycle in G .*

We derive Theorem 4 as a corollary to Theorem 2 as follows. Let G be a graph and $S \subseteq V(G)$. Let \vec{G} be an arbitrary orientation of G and let e_1, \dots, e_m be an enumeration of $E(\vec{G})$. Define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ by $\gamma(e_i) = (2^i, 2^i)$, if e_i has at least one end in S , and $\gamma(e_i) = (0, 0)$, otherwise. Let Z_i be the set of edges of (\vec{G}, γ) that are contained in a Γ_i -zero cycle and suppose C_1 and C_2 are distinct Γ_i -non-zero cycles with $E(C_1) \cap E(C_2) \subseteq Z_i$. Let C'_1 and C'_2 be the corresponding cycles in G . Note that C'_1 and C'_2 both meet S , but they do not share any edges which have an end in S . Therefore, $\gamma_i(C_1) \notin \{\gamma_i(C_2), -\gamma_i(C_2)\}$, and so (\vec{G}, γ) is robust. Clearly, every S -cycle in G corresponds to a (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) , which proves Theorem 4.

Cycles not homologous to zero. Let G be a graph embedded in a (possibly non-orientable) surface Σ and let $\mathcal{H}(\Sigma)$ be the (first) homology group of Σ . A cycle C in G is *not homologous to zero* if it is non-zero in $\mathcal{H}(\Sigma)$. We use the following easy proposition. We give a proof at the end of Section 3.

Proposition 5. *Let G be a graph embedded on a surface Σ and let Γ be the homology group of Σ . Then there exists an orientation \vec{G} of G and a labeling $\gamma : E(\vec{G}) \rightarrow \Gamma$ such that the group-value of every closed walk W in (\vec{G}, γ) is precisely the homology class of W .*

By the previous proposition, there is an orientation \vec{G} of G and a labeling $\gamma : E(\vec{G}) \rightarrow \mathcal{H}(\Sigma) \oplus \mathcal{H}(\Sigma)$, such that a cycle C in G is not homologous to zero if and only if the corresponding cycle in (\vec{G}, γ) is (Γ_1, Γ_2) -non-zero. Therefore, as a corollary to Theorem 1, we obtain the half-integral Erdős-Pósa property for cycles not homologous to zero, which we believe is new.

Theorem 6. *For every integer k , there exists an integer $f(k)$ with the following property. Every graph G embedded in a surface Σ either contains k cycles C_1, \dots, C_k such that each C_i is not homologous to zero and each vertex of G is contained in at most two distinct C_i , or there is a set of at most $f(k)$ vertices of G such that every cycle of $G - X$ is homologous to zero.*

Two constraints. The real power of Theorem 1 and Theorem 2 is that we can prove Erdős-Pósa results for cycles satisfying *two* constraints. We now give some examples of this phenomenon. The first is a recent theorem of Kawarabayashi and Kakimura [12].

Theorem 7 (Odd S -cycles, [12]). *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph G , every set $S \subseteq V(G)$, and $k \geq 1$, either there exist k odd S -cycles such that every vertex is in at most two of them, or there exists a set of at most $f(k)$ vertices intersecting every odd S -cycle.*

We derive Theorem 7 as a corollary to Theorem 1 as follows. Let G be a graph and $S \subseteq V(G)$. Let \vec{G} be an arbitrary orientation of G and let e_1, \dots, e_m be an enumeration of $E(\vec{G})$. Define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ by $\gamma(e_i) = (1, 2^i)$, if e_i has at least one end in S , and $\gamma(e_i) = (1, 0)$, otherwise. Theorem 7 follows by observing that every odd S -cycle in G corresponds to a (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) .

Similarly, we also obtain the half-integral Erdős-Pósa property for odd cycles not homologous to zero and for S -cycles not homologous to zero.

Theorem 8 (Odd cycles not homologous to zero). *For every integer k , there exists an integer $f(k)$ with the following property. Every graph G embedded in a surface Σ either contains k odd cycles C_1, \dots, C_k such that each C_i is not homologous to zero and each vertex of G is contained in at most two distinct C_i , or there is a set of at most $f(k)$ vertices of G meeting all odd cycles in G that are not homologous to zero.*

Theorem 9 (S -cycles not homologous to zero). *For every integer k , there exists an integer $f(k)$ with the following property. For every graph G embedded in a surface Σ and for all $S \subseteq V(G)$, either G contains k cycles C_1, \dots, C_k such that each C_i is an S -cycle not homologous to zero and each vertex of G is contained in at most two distinct C_i , or there is a set of at most $f(k)$ vertices of G meeting all S -cycles in G that are not homologous to zero.*

Remark. The functions in Theorem 6, 8, and 9 do not depend on the surface Σ .

We finish by giving a new example of a (Γ_1, Γ_2) -constrained cycle problem where we obtain the (full) Erdős-Pósa property. Let G be a graph and S_1 and S_2 be subsets of vertices of G (not necessarily disjoint). An (S_1, S_2) -cycle is a cycle in G containing at least one vertex of S_1 and at least one vertex of S_2 . Let \vec{G} be an arbitrary orientation of G and e_1, \dots, e_m be an enumeration of $E(\vec{G})$. Define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z} \oplus \mathbb{Z}$ by $\gamma(e_i) = (\delta_1(e_i)2^i, \delta_2(e_i)2^i)$, where $\delta_j(e_i) = 1$ if e_i has at least one end in S_j , and $\delta_j(e_i) = 0$ if e_i does not have an end in S_j . As in the case of S -cycles, it is easy to check that (\vec{G}, γ) is robust. Since every (S_1, S_2) -cycle in G corresponds to a (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) , we obtain the following theorem.

Theorem 10 (S_1 - S_2 -cycles). *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph G , every $S_1, S_2 \subseteq V(G)$, and every positive integer k , either G has k disjoint (S_1, S_2) -cycles or there exists a set of at most $f(k)$ vertices intersecting every (S_1, S_2) -cycle in G .*

As we have seen above, robustness is a very useful condition with many applications. However, the ‘real’ theorem we prove is Theorem 38, which provides canonical obstructions to the Erdős-Pósa property for (Γ_1, Γ_2) -non-zero cycles. Theorem 38 follows fairly straightforwardly from our refined Flat Wall Theorem.

There are further applications of Theorem 38. For example, we prove that cycles not homologous to zero have the *full* Erdős-Pósa property for graphs embedded on an *orientable* surface (Corollary 41). As we cannot derive these results directly from Theorem 1 or 2, we discuss these applications in Section 10.

It may be possible to prove Erdős-Pósa type results for cycles satisfying more than two constraints by extending our results to $\bigoplus_{i=1}^t \Gamma_i$ -labeled graphs. Let (\vec{G}, γ) be a $\bigoplus_{i=1}^t \Gamma_i$ -labeled graph. A cycle is $(\Gamma_1, \dots, \Gamma_t)$ -non-zero if it is Γ_i -non-zero for all i . The following conjecture would imply the half-integral Erdős-Pósa property for cycles satisfying multiple constraints.

Conjecture 11. *Fix $t \geq 1$ a positive integer and let $\Gamma = \bigoplus_1^t \Gamma_i$ where each Γ_i is a group. Then the set of all $\bigoplus_1^t \Gamma_i$ -labeled graphs has the half-integral Erdős-Pósa property for $(\Gamma_1, \dots, \Gamma_t)$ -non-zero cycles. Moreover, the Erdős-Pósa-function does not depend on the choice of $\Gamma_1, \dots, \Gamma_t$.*

Remark. Let S_1, S_2 and S_3 be subsets of vertices of a graph G . An (S_1, S_2, S_3) -cycle is a cycle in G that uses at least one vertex from each of S_1, S_2 and S_3 . Note that the (full) Erdős-Pósa property does not hold for (S_1, S_2, S_3) -cycles. To see this, let G be a large grid, S_1 be the vertices on the top row, S_2 be the vertices on the rightmost column, and S_3 be the vertices on the bottom row. Therefore, the obvious generalization of robustness does not guarantee the (full) Erdős-Pósa property for $(\Gamma_1, \dots, \Gamma_t)$ -non-zero cycles if $t \geq 3$. This shows that in some sense, Conjecture 11 is best possible.

3 Preliminaries

In this section, we introduce a number of concepts and notation we will need going forward. Given two graphs G and H , we denote by $G \cup H$ the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. Analogously, we denote by $G \cap H$ the graph with vertices $V(G) \cap V(H)$ and edges $E(G) \cap E(H)$.

3.1 Group-labeled graphs

In the introduction we claimed that if C_1 and C_2 are cycles such that $E(C_1) = E(C_2)$, then C_1 is Γ -zero if and only if C_2 is Γ -zero. While this is easy to see if Γ is abelian, this is not obvious for non-abelian groups, so we give a proof now. Let $C = v_0 e_1 v_1 e_2 v_2 \dots e_\ell v_\ell$, where $v_0 = v_\ell$. Suppose that C_1 is Γ -zero. Observe that

$$\begin{aligned} 0 &= \gamma(e_1, v_1) + \dots + \gamma(e_\ell, v_\ell) \\ &= -(\gamma(e_1, v_1) + \dots + \gamma(e_\ell, v_\ell)) \\ &= ((-\gamma(e_\ell, v_\ell)) + \dots + (-\gamma(e_1, v_1))) \\ &= \gamma(e_\ell, v_{\ell-1}) + \dots + \gamma(e_1, v_0). \end{aligned}$$

Thus, the cycle starting at v_0 but with the opposite orientation as C also has group-value 0. Furthermore,

$$\begin{aligned} 0 &= \gamma(e_1, v_1) + \dots + \gamma(e_\ell, v_\ell) \\ &= -\gamma(e_1, v_1) + (\gamma(e_1, v_1) + \dots + \gamma(e_\ell, v_\ell)) + \gamma(e_1, v_1) \\ &= \gamma(e_2, v_2) + \dots + \gamma(e_\ell, v_\ell) + \gamma(e_1, v_1). \end{aligned}$$

Thus, the cycle with the same orientation as C but starting at v_1 also has group-value 0. By induction, every cycle with the same edge set as C also has group-value 0, as claimed.

In a slight abuse of notation, we will use C to also refer to the graph with vertex set $V(C)$ and edge set $E(C)$, depending on the context.

Let $v \in V(G)$, $\alpha \in \Gamma$, and (\vec{G}', γ') be the Γ -labeled graph obtained from (\vec{G}, γ) by adding α (on the right) to the labels of the edges with head v and adding $-\alpha$ (on the left) to the labels of the edges with tail v . We refer to this operation as *shifting at v (by α)*. A Γ -labeled graph that can be obtained from (\vec{G}, γ) via a sequence of shifts is said to be *shifting-equivalent* to (\vec{G}, γ) . We note that a key property of shifting is that it does change the set of Γ -non-zero cycles (if Γ is abelian, then the group-value of each cycle is actually unaltered by shifting). In addition, the robustness of a group-labeling is maintained when performing shifts.

3.2 Minors in graphs and the Flat Wall Theorem

A *separation* in a graph G is a pair (A, B) where A and B are edge-disjoint subgraphs of G such that $A \cup B = G$. The *order* of a separation (A, B) is $|V(A) \cap V(B)|$. We say that (A, B) is a *k -separation* if it has order at most k . The separation is *trivial* if $V(A) \subseteq V(B)$ or $V(B) \subseteq V(A)$.

Let $k \geq 1$ be a positive integer and let G be a graph. A *tangle of order k* in G is a set \mathcal{T} of $(k-1)$ -separations (A, B) which satisfy the following.

- (T1) For every $(k-1)$ -separation (A, B) of G , either $(A, B) \in \mathcal{T}$ or $(B, A) \in \mathcal{T}$;
- (T2) $V(A) \neq V(G)$ for all $(A, B) \in \mathcal{T}$;
- (T3) $A_1 \cup A_2 \cup A_3 \neq G$ for all $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$.

If $(A, B) \in \mathcal{T}$, we call A the \mathcal{T} -*small* side of the separation. A separation and a tangle in a group-labeled graph (\vec{G}, γ) is simply a separation and a tangle in G , respectively.

An example of a tangle which we will use going forward is the following tangle defined on a clique.

Lemma 12 ([17]). *Let $t = \lceil \frac{2n}{3} \rceil$ and \mathcal{T} be the set of all $(t-1)$ -separations (A, B) of K_n such that $V(B) = V(K_n)$. Then \mathcal{T} is a tangle.*

We now describe several additional tangle constructions. Let \mathcal{T} be a tangle of order k in G and let $\ell \leq k$. Define \mathcal{T}' to be the set of ℓ -separations (A, B) such that $(A, B) \in \mathcal{T}$. It is immediate that \mathcal{T}' is also a tangle, which we call a *truncation* or *restriction* of \mathcal{T} . Note that \mathcal{T}' is a truncation of \mathcal{T} if and only if $\mathcal{T}' \subseteq \mathcal{T}$.

Let G be a graph and $e \in E(G)$. We let $G \setminus e$ be the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e\}$. We say $G \setminus e$ is the graph obtained from G by *deleting* e . On the other hand, G/e is the graph obtained from G by deleting e and then identifying the endpoints of e . We say G/e is the graph obtained from G by *contracting* e . If $x \in V(G)$, we let $G - x$ be the graph obtained from G by deleting x and deleting all edges that have x as an endpoint. Note that deletion and contraction of edges commute and do not depend on the order

in which they are performed. Thus, if C and D are disjoint subsets of edges, we let $G/C \setminus D$ be the graph obtained from G by deleting all edges in D and contracting all edges in C .

A graph H is a *minor* of a graph G if a graph isomorphic to H can be obtained from G by deleting edges, contracting edges, and deleting vertices. Equivalently, H is a minor of G if there is a function π from $V(H) \cup E(H)$ to subgraphs of G such that

- (i) $\pi(v)$ is a tree and a subgraph of G and $\pi(v)$ is disjoint from $\pi(u)$ for distinct $u, v \in V(H)$; and
- (ii) $\pi(uv)$ is an edge in G joining $\pi(u)$ and $\pi(v)$ for all $uv \in E(H)$.

We say that π is an H -model in G .

We next describe how to obtain a tangle in a graph from a tangle in one of its minors. Let H be a minor of G and suppose that \mathcal{T} is a tangle of order $k \geq 2$ in H . Let C and D be disjoint subsets of edges and $X \subseteq V(G)$ be such that $(G/C \setminus D) - X$ is isomorphic to H . If A is a subgraph of G we let $C_A := C \cap A$, $D_A := D \cap A$ and $X_A := X \cap A$. Define \mathcal{T}_H to be the set of $(k-1)$ -separations (A, B) of G such that $((A/C_A \setminus D_A) - X_A, (B/C_B \setminus D_B) - X_B) \in \mathcal{T}$. It follows (see [17]) that \mathcal{T}_H is a tangle in G . We say that \mathcal{T}_H is the tangle in G induced by \mathcal{T} . If K is a clique-minor in G we always let \mathcal{T}_K denote the tangle in G induced by the tangle in K from Lemma 12.

If \mathcal{T} is tangle of order k in G and X is a subset of vertices of size at most $k-2$, then it is easy to show that there is a unique block U of $G-X$ such that $V(U) \cup X$ is not contained in any \mathcal{T} -small side. We call U the \mathcal{T} -large block of $G-X$.

Tangles are extremely useful objects in graph structure theory. For example, they arise in a natural way when considering the Erdős-Pósa-property. Let Γ_1 and Γ_2 be groups, \mathcal{G} a set of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs, and $f : \mathbb{N} \rightarrow \mathbb{N}$ a function. A pair $((\vec{G}, \gamma), k)$ is a *minimal counterexample to f being an Erdős-Pósa function* if it satisfies the following conditions:

- (MC1) (\vec{G}, γ) is an element of \mathcal{G} ;
- (MC2) there does not exist a set $X \subseteq V(G)$ with $|X| \leq f(k)$ such that $(\vec{G} - X, \gamma)$ has no (Γ_1, Γ_2) -non-zero cycle, nor does (\vec{G}, γ) contain k disjoint (Γ_1, Γ_2) -non-zero cycles;
- (MC3) for all $(\vec{G}', \gamma') \in \mathcal{G}$ and for all $k' < k$, the graph (\vec{G}', γ') either has k' disjoint (Γ_1, Γ_2) -non-zero cycles or there exists $X' \subseteq V(\vec{G}')$ with $|X'| \leq f(k')$ such that $(\vec{G}' - X', \gamma')$ has no (Γ_1, Γ_2) -non-zero cycle.

The definition for a pair $((\vec{G}, \gamma), k)$ being a minimal counterexample to f being a half-integral Erdős-Pósa function is analogous.

The following result is Lemma 2.1 of [20], rephrased in terms of (Γ_1, Γ_2) -group-labeled graphs. We include the proof for completeness.

Lemma 13 (Lemma 2.1 [20]). *Let Γ_1, Γ_2 be groups and \mathcal{G} a set of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function that is not an Erdős-Pósa function (not a half-integral Erdős-Pósa-function) for (Γ_1, Γ_2) -non-zero cycles.*

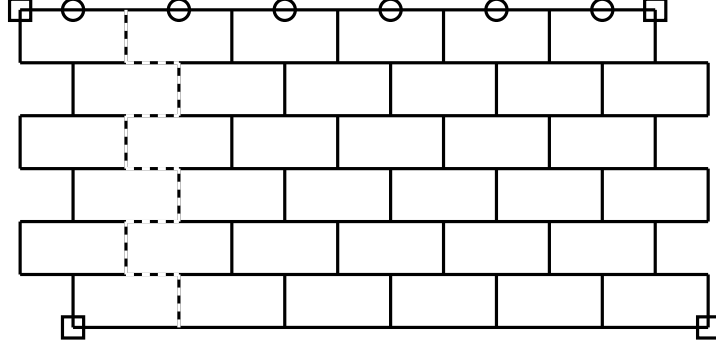


Figure 1: An elementary 6-wall W . The nails of W on the top row are circled, the corners are marked by squares, and the second vertical path is marked dashed.

Let $((\vec{G}, \gamma), k)$ be a minimal counterexample to f being an Erdős-Pósa function (half-integral Erdős-Pósa-function). Let $t \in \mathbb{N}$ be such that $t \leq f(k) - 2f(k-1)$ and $t \leq f(k)/3$, and let \mathcal{T} be the set of all $(t-1)$ -separations (A, B) of (\vec{G}, γ) such that B contains a (Γ_1, Γ_2) -non-zero cycle. Then \mathcal{T} is a tangle of order t in (\vec{G}, γ) .

Proof. We prove the statement for when f is not an Erdős-Pósa function; the proof for the case when f is not a half-integral Erdős-Pósa function is identical.

Let (A, B) be a $(t-1)$ -separation in (\vec{G}, γ) . We claim that exactly one of A or B contains a (Γ_1, Γ_2) -non-zero cycle. If neither A nor B contains a (Γ_1, Γ_2) -non-zero cycle, then $V(A) \cap V(B)$ is a set intersecting every such cycle, a contradiction to (MC2). Assume now that both A and B contain a (Γ_1, Γ_2) -non-zero cycle. Then neither of the $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs $A - V(B)$ and $B - V(A)$ contain $k-1$ disjoint (Γ_1, Γ_2) -non-zero cycles. By (MC3), there exist vertex sets X_A and X_B each of size at most $f(k-1)$ which intersect every (Γ_1, Γ_2) -non-zero cycle in $A - V(B)$ and $B - V(A)$, respectively. Thus $X = X_A \cup X_B \cup (V(A) \cap V(B))$ is a set intersecting every (Γ_1, Γ_2) -non-zero cycle in (\vec{G}, γ) , a contradiction to (MC2). We conclude that exactly one of A and B contains a (Γ_1, Γ_2) -non-zero cycle, as claimed.

It follows that (T1) and (T2) hold for \mathcal{T} . To see that (T3) holds as well, assume there exist separations (A_1, B_1) , (A_2, B_2) , and (A_3, B_3) in \mathcal{T} with $(\vec{G}, \gamma) = A_1 \cup A_2 \cup A_3$. Since A_i contains no (Γ_1, Γ_2) -non-zero cycle, every cycle in $A_1 \cup A_2 \cup A_3 = (\vec{G}, \gamma)$ must meet $X := \bigcup_{i=1}^3 (V(A_i) \cap V(B_i))$. Since X is of size at most $3t-3$, we contradict (MC2). This completes the proof. \square

Theorem 22 is a refinement of the Flat Wall Theorem of Robertson and Seymour [18, Theorem 9.8]. We first need some definitions and notation before we can state the Flat Wall Theorem.

We first define an elementary r -wall. An *elementary 6-wall* is shown in Figure 1. Let $r, s \geq 2$ be an integer. An $r \times s$ -grid is the graph with vertex set $[r] \times [s]$ in which (i, j) is adjacent to (i', j') if and only if $|i - i'| + |j - j'| = 1$. An *elementary r -wall* is obtained from the $2(r+1) \times (r+1)$ -grid by deleting all edges with ends $(2i-1, 2j-1)$ and $(2i-1, 2j)$ for all $i \in [r]$ and $j \in [\lceil r/2 \rceil]$ and all edges

with ends $(2i, 2j)$ and $(2i, 2j + 1)$ for all $i \in [r]$ and $j \in \lfloor (r-1)/2 \rfloor$ and then deleting the two resulting vertices of degree 1. Every facial cycle of a finite face has length 6, and is called a *brick*. The four vertices that are the intersection of a leftmost or rightmost vertical path with a topmost or bottommost horizontal path are called the *corners* of the wall. We denote the vertices of degree 2 which are not corners as the *nails* of the wall.

A subdivision of an elementary t -wall is called a t -wall or simply a *wall*. The *bricks* of a wall are the subdivided 6-cycles corresponding to the bricks of the elementary wall. Similarly, the *corners* and *nails* of a wall are the vertices corresponding to the corners and nails of the elementary wall before subdividing edges.

Sometimes we tacitly assume that a wall is embedded in the plane as shown in Figure 1. The *outercycle* or *boundary cycle* of the wall is the facial cycle of the infinite face. Let W be a t -wall. Let v_1, v_2, v_3 , and v_4 be the four corners of the wall as they occur in that clockwise order on the boundary cycle C with v_1 the vertex corresponding to the corner $(1, 2)$ of the elementary t -wall. Let $P_0^{(h)}$ be the subpath of C with endpoints v_1 and v_2 which is disjoint from $\{v_3, v_4\}$ and let $P_t^{(h)}$ be the subpath of C with ends v_3 and v_4 which is disjoint from $\{v_1, v_2\}$. Observe that there is a unique set of disjoint paths $P_0^{(v)}, \dots, P_t^{(v)}$ such that each path has one end in $V(P_0^{(h)})$, the other end in $V(P_t^{(h)})$, and no other vertex in $V(P_0^{(h)} \cup P_t^{(h)})$. We call $P_0^{(v)}, \dots, P_t^{(v)}$ the *vertical paths* of W . There is also a unique set of disjoint paths $P_1^{(h)}, \dots, P_{t-1}^{(h)}$, where each has one end in $P_0^{(v)}$, the other end in $P_t^{(v)}$, and no other vertex in $P_0^{(v)} \cup P_t^{(v)} \cup P_0^{(h)} \cup P_t^{(h)}$. The paths $P_0^{(h)}, \dots, P_t^{(h)}$ are the *horizontal paths* of the wall. Note that if we are just given W as a graph, then the nails and corners are not necessarily uniquely defined; in such cases, we assume that the nails and corners are arbitrarily chosen.

Next, we define a subwall of a t -wall W . Let $s \leq t$. An s -subwall of W is a subgraph W' of W such that W' is an s -wall and there exists a choice of corners of W' so that every horizontal path of W' is a subpath of a horizontal path of W and every vertical path of W' is a subpath of a vertical path of W . Let $I^{(h)}, I^{(v)} \subseteq \{0, \dots, t\}$ with $|I^{(h)}| = |I^{(v)}| = s + 1$ and such that every horizontal path of W' is a subpath of $P_j^{(h)}$ for some $j \in I^{(h)}$ and every vertical path of W' is a subpath of $P_j^{(v)}$ for some $j \in I^{(v)}$. We say that W' is k -contained in W if $\min\{I^{(h)}, I^{(v)}\} \geq k$ and $\max\{I^{(h)}, I^{(v)}\} \leq t - k$.

We observed that there may be several possible choices for the set of nails of a wall W . However, if W' is a 1-contained subwall of W , there are choices of nails N' for W' which are more natural than others; that is, we assume that

$$N' \subseteq \{v \in V(C') : d_{W'}(v) = 2 \text{ and } d_W(v) = 3\} \setminus \{v'_1, \dots, v'_4\},$$

where C' is the outercycle of W' and v'_1, \dots, v'_4 are the corners of W' . We say that N' is the set of *nails of W' with respect to W* .

Let us turn to the definitions needed for the Flat Wall Theorem.

Definition 14. Let G be a graph, $X \subseteq V(G)$, and (C, D) be a separation of G such that

$$(i) \quad |V(C \cap D)| \leq 3,$$

$$(ii) \quad X \subseteq V(C), \text{ and}$$

- (iii) there is a family of $|V(C \cap D)|$ paths from a vertex $d \in D - C$ to X that are disjoint except for d .

Let H be the graph obtained from C by adding exactly those edges such that $C \cap D$ is a clique. We say that H is an **elementary X -reduction** of G (with respect to (C, D)). An **X -reduction** of G is a graph that can be obtained from G via a sequence of elementary X -reductions.

Definition 15. Let G be a graph and let W be a wall in G with outercycle O . Let (A, B) be a separation of G such that

- (i) $V(A \cap B) \subseteq V(O)$, and $V(W) \subseteq V(B)$,
- (ii) there is a choice of nails and corners of W such that every nail and corner is in A , and
- (iii) there is a $V(A \cap B)$ -reduction G_0 of B such that G_0 can be drawn in a disk Δ with all vertices of $A \cap B$ drawn on the boundary of Δ according to their order on O .

In this case we say that the wall W is **flat** in G . The separation (A, B) **certifies** that W is flat.

Let W be a t -wall. We now describe a tangle \mathcal{T} of order $t + 1$ inside W as follows. Let \mathcal{T} consist of the set of all t -separations (A, B) such that B contains a horizontal path. The set \mathcal{T} is indeed a tangle (see [17]). If W is a minor of G , we always let \mathcal{T}_W denote the tangle in G induced by the tangle in W described above. We can now state the Flat Wall Theorem.

Theorem 16 (Theorem 1.5 [13]). Let $r, t \in \mathbb{N}$, $R := 49152t^{24}(40t^2 + r)$, G be a graph, and W be an R -wall in G . Then either G has a K_t -minor K such that \mathcal{T}_K is a restriction of \mathcal{T}_W , or there exists a subset Z of vertices of G of size at most $12288t^{24}$ and an r -subwall W' of W such that $V(W') \cap Z$ is empty and W' is flat in $G - Z$.

Thus, loosely speaking, every large wall W in a graph G can be used to construct a large clique minor, or alternatively, we can delete a bounded size set of vertices such that a large subwall of W induces a planar subgraph of G up to 1-, 2-, and 3-sums.

Observe that a subwall of a flat wall is once again flat and thus we also speak about *flat subwalls*.

Let (\vec{G}, γ) be a Γ -labeled graph. We abuse terminology slightly and call a subgraph (\vec{W}, γ) of (\vec{G}, γ) a *wall* if W is a wall in G . We naturally extend all terminology for walls in graphs to walls in group-labeled graphs.

We say that (\vec{G}, γ) is *null-labeled* if for all $e \in E(\vec{G})$, we have $\gamma(e) = 0$. We say that (\vec{G}, γ) is Γ -*bipartite* if every cycle of (\vec{G}, γ) is Γ -zero. If $\Gamma = \Gamma_1 \oplus \Gamma_2$, then (\vec{G}, γ) is Γ_i -*bipartite* if every cycle of (\vec{G}, γ_i) is Γ_i -zero.

We will repeatedly use the following basic lemma [8] without explicit reference.

Lemma 17 ([8]). Let Γ be a group and (\vec{G}, γ) be a Γ -labeled graph. If (\vec{G}, γ) is Γ -bipartite, then (\vec{G}, γ) is shifting-equivalent to a null-labeled graph.

We say that a Γ -labeled graph (\vec{H}, γ_H) is a *minor* of another Γ -labeled graph (\vec{G}, γ_G) if (\vec{H}, γ_H) can be obtained from (\vec{G}, γ_G) via any combination of edge deletions, vertex deletions, shifts, and contracting null-labeled edges. Note that the definition of contracting an edge does not change the orientations of any other edge. In other words, if we contract a null-labeled edge e to obtain a new vertex v_e , and f is an edge in (\vec{G}, γ) with $\text{tail}_{\vec{G}}(f)$ (respectively, $\text{head}_{\vec{G}}(f)$) equal to an end of e in \vec{G} , then $\text{tail}_{\vec{G}/e}(f) = v_e$ (respectively, $\text{head}_{\vec{G}/e}(f) = v_e$).

3.3 Graphs and homology

As promised, we finish this section by proving Proposition 5.

Proof of Proposition 5. Let G be a graph embedded on a surface Σ and Γ be the homology group of Σ . We proceed by induction on $|E(G)|$. For a closed walk W in G , we let $h(W)$ denote the homology class of W . Let $e = uv \in E(G)$. By induction, there is an orientation $\vec{G - e}$ of $G - e$ and a labeling $\gamma : E(\vec{G - e}) \rightarrow \Gamma$ such that $\gamma(W)$ is the homology class of W for all closed walks W in $G - e$. We orient uv from u to v . If there is no path from v to u in $G - e$ we let $\gamma(uv)$ be an arbitrary element of Γ . Otherwise, we choose a path P in $G - e$ from v to u , and we let C be the cycle in G obtained by following P and then uv . We then define $\gamma(uv)$ to be $h(C) - \gamma(P)$. Suppose we are in the first case. Let W be a closed walk in G . We may assume that $uv \in W$, otherwise there is nothing to prove. We may also assume that W begins and ends at u . Thus, $W = uvW_1vuW_2uv \dots W_{2k-1}vuW_{2k}$, where each W_i is a closed walk in $G - uv$ (possibly W_i is a walk of length 0). By induction, we have $\gamma(W_i) = h(W_i)$ for all $i \in [k]$. Finish by observing that $\gamma(W) = \sum_{i=1}^{2k} \gamma(W_i) = \sum_{i=1}^{2k} h(W_i) = h(W)$. In the second case, let W be a closed walk in G with $uv \in W$. It suffices to consider the case that $W = uvQ$, where Q is a walk from v to u in $G - uv$. Let $W' = (uvP)(P^{-1}Q)$, where P is the path used to define $\gamma(uv)$. Since $P^{-1}Q$ is a walk in $G - uv$, by induction we have $h(P^{-1}Q) = -\gamma(P) + \gamma(Q)$. Thus, $h(W) = h(W') = h(uvP) + h(P^{-1}Q) = \gamma(uv) + \gamma(P) - \gamma(P) + \gamma(Q) = \gamma(W)$. \square

4 Obstructions

In this section we introduce some canonical counterexamples to the Erdős-Pósa property for (Γ_1, Γ_2) -non-zero cycles that appear in Theorem 38.

In his proof of Theorem 3, Reed [16] proves a significantly stronger statement. He defines a set of canonical counterexamples to the Erdős-Pósa property for odd cycles called *Escher-walls* and then shows that any graph either has many disjoint odd cycles, or a bounded set of vertices intersecting every odd cycle, or it contains a large Escher-wall. An *Escher-wall of height h* , is a graph obtained from a bipartite h -wall W by adding h disjoint paths P_1, \dots, P_h such that for all $i \in [h]$,

- (i) P_i has one end in the i -th brick of the top horizontal path of W and the other end in the $(h - i + 1)$ -th brick of the bottom horizontal path of W , but otherwise contains no other vertices of W , and
- (ii) $W \cup P_i$ contains an odd cycle.

It is easy to show that an Escher-wall G of height h does not contain two disjoint odd cycles nor a set $X \subseteq V(G)$ with $|X| < h$ and $G - X$ bipartite. Reed [16] showed that Escher-walls are the only obstructions to the Erdős-Pósa property for odd cycles.

Theorem 18 ([16]). *There exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph G and every integer k ,*

- (i) G contains k disjoint odd cycles,
- (ii) there exists a set X such that $G - X$ is bipartite and $|X| \leq f(k)$, or
- (iii) G contains an Escher-wall of height k .

We will prove a similar result (Theorem 38) for (Γ_1, Γ_2) -non-zero cycles in Section 10, which in fact will imply Theorem 18.

We begin with some definitions. Let $t \geq 1$ be an integer and W be a t -wall with a set of nails N . Let $P_j^{(h)}$ and $P_j^{(v)}$ for $0 \leq j \leq t$ be the horizontal and vertical paths of W . For any subwall W' of W with set of nails N' , let j be the minimum index such that $P_j^{(h)}$ intersects W' , and let P be the horizontal path of W' contained in $P_j^{(h)}$. We define the *top nails* of W' to be the linearly ordered set of vertices \overline{N} consisting of the nails of W' contained in P such that $x, y \in \overline{N}$ satisfy $x \leq y$ if and only if we encounter x before y when traversing $P_j^{(h)}$ from its endpoint in $P_0^{(v)}$ to its endpoint in $P_t^{(v)}$.

Let G be a graph. For $X, Y \subseteq V(G)$ an X - Y -path is a path P with at least one edge such that one end of P is in X the other end is in Y , and $V(P)$ is otherwise disjoint from $X \cup Y$. An X -path is an X - X -path. An X -linkage (or simply a *linkage*) is a family of disjoint X -paths in G . For a subgraph H of G , a H -linkage is a set of pairwise disjoint $V(H)$ -paths which are edge-disjoint from H .

Suppose in addition that the vertex set X is a linearly ordered set. If P is an X -path, we call the smaller endpoint of P the *left endpoint*, and the larger endpoint of P the *right endpoint*. Let $x, y \in X$ with $x \leq y$. We let $[x, y]$ denote the set of all $z \in X$ such that $x \leq z \leq y$, and call $[x, y]$ an *interval*. We write $[x, y] < [z, w]$ if $y < z$. For a family \mathcal{P} of disjoint X -paths, we define $I_{\mathcal{P}} \subseteq X$ as the minimal interval (under inclusion) containing all endpoints of paths in \mathcal{P} . We let $I_{\mathcal{P}}^{\ell}$ and $I_{\mathcal{P}}^r$ be the minimal intervals containing all the left and right endpoints of \mathcal{P} , respectively.

In the following we tacitly assume that X -paths are always traversed from their left endpoint to their right endpoint. Let (p_1, p_2) and (q_1, q_2) be pairs of elements of X such that $p_1 < q_1$, $p_1 < p_2$, and $q_1 < q_2$. We say that (p_1, p_2) and (q_1, q_2) are *in series* if $p_2 < q_1$; *nested* if $p_1 < q_1 < q_2 < p_2$; or *crossing* if $p_1 < q_1 < p_2 < q_2$.

Let P and Q be disjoint X -paths with ends (p_1, p_2) and (q_1, q_2) , respectively. We say that P and Q are *in series*, *nested*, or *crossing*, according as (p_1, p_2) and (q_1, q_2) are in series, nested, or crossing. A collection \mathcal{P} of X -paths is *in series*, *nested*, or *crossing* if all pairs of paths in \mathcal{P} are in series, nested, or crossing, respectively. We say that \mathcal{P} is *pure* if \mathcal{P} is nested, crossing, or in series. Thus, there are three *types* of pure linkages. See Figure 2 for illustration.

We now describe our set of obstructions. Let Γ_1 and Γ_2 be non-trivial groups. Let h be a positive integer and let (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph be such that

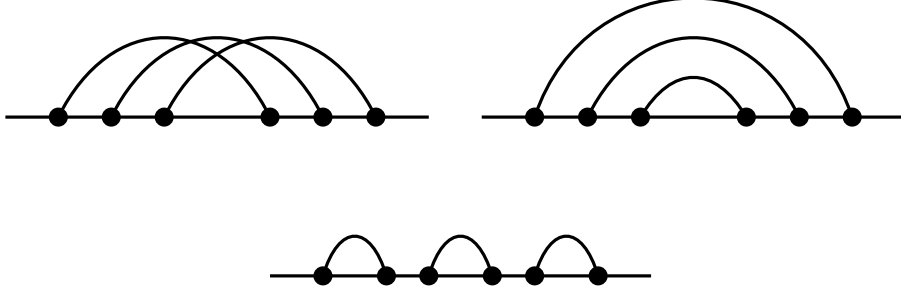


Figure 2: Linkages of size 3 that are crossing, nested, and in series, respectively. The six endpoints of the paths are linearly ordered from left to right.

- (i) $E(G) = E(W) \cup E(\mathcal{P}) \cup E(\mathcal{Q})$, where W is a $4h$ -wall, and \mathcal{P} and \mathcal{Q} are pure W -linkages of size h such that the endpoints of the paths in \mathcal{P} or \mathcal{Q} are vertices in the topmost horizontal path of W of degree 2 in W , no two of which lie in the same brick of W , and are not corners of W ,
- (ii) (\vec{W}, γ) is null-labeled,
- (iii) $\mathcal{P} \cup \mathcal{Q}$ is a linkage of size $2h$,
- (iv) \mathcal{P} and \mathcal{Q} are of different type,
- (v) $\gamma_1(P) \neq 0$ and $\gamma_2(P) = 0$ for all $P \in \mathcal{P}$,
- (vi) $\gamma_1(Q) = 0$ and $\gamma_2(Q) \neq 0$ for all $Q \in \mathcal{Q}$,
- (vii) if \mathcal{P} is crossing or nested, then $\gamma_1(P_1) = \gamma_1(P_2)$ for all $P_1, P_2 \in \mathcal{P}$,
- (viii) if \mathcal{Q} is crossing or nested, then $\gamma_2(Q_1) = \gamma_2(Q_2)$ for all $Q_1, Q_2 \in \mathcal{Q}$,
- (ix) If $I_{\mathcal{P}} \cap I_{\mathcal{Q}} = \emptyset$, then $I_{\mathcal{P}} < I_{\mathcal{Q}}$ and otherwise if $I_{\mathcal{P}} \cap I_{\mathcal{Q}} \neq \emptyset$, then $I_{\mathcal{P}}^{\ell} < I_{\mathcal{Q}}^{\ell} < I_{\mathcal{P}}^r < I_{\mathcal{Q}}^r$ and neither \mathcal{P} nor \mathcal{Q} is in series.

It is not difficult to see that (\vec{G}, γ) does not contain two disjoint (Γ_1, Γ_2) -non-zero cycles. However, one needs more than h vertices to cover all (Γ_1, Γ_2) -non-zero cycles in (\vec{G}, γ) .

Remark. Note that if \mathcal{P} is nested and \mathcal{Q} is in series, then (\vec{G}, γ) is planar. Thus, (Γ_1, Γ_2) -non-zero cycles do not have the (full) Erdős-Pósa property even when restricted to $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs (\vec{G}, γ) for which G is planar.

5 Γ -Odd K_t -models

Let Γ be a group and (\vec{G}, γ) be a Γ -labeled graph. A K_t -model π in (\vec{G}, γ) is simply a K_t -model in G . That is, π is a function mapping $V(K_t) \cup E(K_t)$ to subgraphs of (\vec{G}, γ) such that

- (i) $\{\pi(v)\}_{v \in V(K_t)}$ is a set of pairwise disjoint trees in (\vec{G}, γ) , and

- (ii) for all $uv \in E(K_t)$, $\pi(uv)$ is an edge of (\vec{G}, γ) joining $\pi(u)$ and $\pi(v)$.

A K_t -model in (\vec{G}, γ) is Γ -odd if it additionally satisfies

- (iii) for all distinct $x, y, z \in V(K_t)$, the unique cycle of (\vec{G}, γ) in $\pi(x) \cup \pi(y) \cup \pi(z) \cup \pi(xy) \cup \pi(xz) \cup \pi(yz)$ is Γ -non-zero.

Observe that if $\Gamma = \mathbb{Z}_2$, then (\vec{G}, γ) has a Γ -odd K_t -model if and only if (\vec{G}, γ) has an odd- K_t -minor in the sense of [9].

Next, we extend this definition to the case when G is a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. A (Γ_1, Γ_2) -odd K_t -model in (\vec{G}, γ) is a function π mapping $V(K_t) \cup E(K_t)$ to subgraphs of (\vec{G}, γ) such that

- (i) $\{\pi(v)\}_{v \in V(K_t)}$ is a set of pairwise disjoint trees in (\vec{G}, γ) ,
- (ii) for all $uv \in E(K_t)$, $\pi(uv)$ is a set of either one or two edges of (\vec{G}, γ) such that for all $e \in \pi(uv)$, e joins $\pi(u)$ and $\pi(v)$, and
- (iii) for all $i \in [2]$ and distinct $x, y, z \in V(K_t)$, there are $e_{xy} \in \pi(xy), e_{xz} \in \pi(xz), e_{yz} \in \pi(yz)$ such that the unique cycle of $\pi(x) \cup \pi(y) \cup \pi(z) \cup e_{xy} \cup e_{xz} \cup e_{yz}$ is Γ_i -non-zero.

Let π be a K_t -model in (\vec{G}, γ) . We say $V(K_t)$ and $E(K_t)$ are the *vertices* and *edges* of π , respectively. Recall that there is a tangle of order $\lceil \frac{2t}{3} \rceil$ in K_t (see Lemma 12). Thus, π induces a tangle of order $\lceil \frac{2t}{3} \rceil$ in (\vec{G}, γ) , which we denote as \mathcal{T}_π . We say π' is an *enlargement* of π if π' is a $K_{t'}$ -model in (\vec{G}, γ) , and for all $v \in V(K_{t'})$, the tree $\pi'(v)$ contains some $\pi(u)$ for $u \in V(K_t)$ (and thus $t' \leq t$).

The following lemma will be used frequently in our proofs.

Lemma 19 ([3]). *Let Γ be a group and (\vec{G}, Γ) be a Γ -labeled graph. For every $A \subseteq V(G)$, one of the following two statements hold.*

- (i) *there are k pairwise disjoint Γ -non-zero A -paths in (\vec{G}, γ) , or*
- (ii) *there is a set X of at most $2k - 2$ vertices such that $(\vec{G} - X, \gamma)$ does not contain a Γ -non-zero A -path.*

Let $R(n, m)$ be the least number such that for every $\{\text{red, blue}\}$ -colouring of the edges of the 3-uniform complete hypergraph on $R(n, m)$ vertices, there is always a red 3-uniform complete hypergraph on n vertices or a blue 3-uniform complete hypergraph on m vertices. It is well known that $R(n, m)$ is finite for all positive integers n, m .

Lemma 20. *Let Γ be a group and (\vec{G}, γ) be a Γ -labeled graph. For every $K_{R(t, 10t)}$ -model π in (\vec{G}, γ) , there is either a Γ -odd K_t -model in (\vec{G}, γ) that is an enlargement of π , or there is a set of vertices X such that $|X| < 8t$ and the \mathcal{T}_π -large block of $(\vec{G} - X, \gamma)$ is Γ -bipartite.*

Proof. Let π be a $K_{R(t,10t)}$ -model in a Γ -labeled graph (\vec{G}, γ) . Let H be a 3-uniform complete hypergraph with vertex set $\bigcup_{u \in V(K_{R(t,10t)})} \pi(u)$. Color an edge xyz of H red, if the cycle of (\vec{G}, γ) in

$$\pi(x) \cup \pi(y) \cup \pi(z) \cup \pi(xy) \cup \pi(xz) \cup \pi(yz);$$

is Γ -non-zero; otherwise, color xyz blue. By the definition of the Ramsey number $R(t, 10t)$, there is a Γ -odd K_t -model in (\vec{G}, γ) or a Γ -bipartite K_{10t} -model η in (\vec{G}, γ) that is an enlargement of π .

Henceforth, we assume the latter case. Let $\{v_1, \dots, v_{10t}\}$ be the vertices of η . Since η is Γ -bipartite, by shifting we may assume that all edges of η are null-labeled (the definition of a Γ -odd K_t -model is invariant under shifting).

Next we show the following claim.

Suppose there are t disjoint Γ -non-zero paths P_1, \dots, P_t in (\vec{G}, γ) such that for all $i \in [t]$, the path P_i is a $\eta(v_{2i-1})$ - $\eta(v_{2i})$ -path and $\eta(v_j) \cap P_i = \emptyset$ for all $j \in [2t] \setminus \{2i-1, 2i\}$, then there is a Γ -odd K_t -model η' in (\vec{G}, γ) which is an enlargement of η (and thus of π). (1)

Suppose there are t paths P_1, \dots, P_t as in (1). Without loss of generality, we may assume that the edge $\eta(v_i v_j)$ is directed towards v_j for all $i, j \in [t]$, $i < j$. Let $\{w_1, \dots, w_t\}$ be the vertices of the following defined K_t -model η' . Define $\eta'(w_i) = \eta(v_{2i-1}) \cup \eta(v_{2i}) \cup E(P_i)$ and $\eta'(w_i w_j) = \eta(v_{2i} v_{2j-1})$ for $j > i$.

Let $i < j < k \leq t$ and let C be the unique cycle in

$$\eta'(w_i) \cup \eta'(w_j) \cup \eta'(w_k) \cup \eta'(w_i w_j) \cup \eta'(w_i w_k) \cup \eta'(w_j w_k).$$

Since $E(P_j) \subseteq E(C)$, $\gamma(P_j) \neq 0$, and all other edges of C are null-labeled, we have $\gamma(C) \neq 0$. Thus, η' is a Γ -odd K_t -model in (\vec{G}, Γ) which is an enlargement of η , as required.

Let us come back to the Γ -bipartite K_{10t} -model η . We now show that there is a collection of paths as described in (1) or a set X such that $|X| < 8t$ and the \mathcal{T}_π -large block of $(\vec{G} - X, \gamma)$ is Γ -bipartite. Recall that $V(\eta) = \{v_1, \dots, v_{10t}\}$. Pick from each tree $\eta(v_i)$ a vertex s_i and let $S = \{s_1, \dots, s_{10t}\}$.

By Lemma 19, there is a set X such that $|X| \leq 8t - 2$ and $(\vec{G} - X, \gamma)$ does not contain a Γ -non-zero S -path or (\vec{G}, Γ) contains $4t$ disjoint Γ -non-zero S -paths. Suppose such a set X exists. Since $|X| < 8t$, we may assume that $\eta(v_i) \cap X = \emptyset$ for $i \leq 2t$.

Let (\vec{U}, γ) be the \mathcal{T}_η -large block of $(\vec{G} - X, \gamma)$. Assume for a contradiction that (\vec{U}, γ) is not Γ -bipartite and hence contains a Γ -non-zero cycle C . For $i \in [2]$, if $s_i \notin V(U)$, then let x_i be the unique vertex in $U \cap \eta(v_i)$ such that there is a s_i - x_i -path Q_i internally-disjoint from U in $\eta(v_i)$. If $s_i \in V(U)$, set $x_i = s_i$. Since U is 2-connected, there are two disjoint paths Q'_1, Q'_2 joining $\{x_1, x_2\}$ and C . Since C is a Γ -non-zero cycle, there is a Γ -non-zero s_1 - s_2 -path in $(\vec{G} - X, \gamma)$ by combining Q_1, Q'_1 , a suitable part of the cycle C , Q'_2 , and Q_2 . This contradicts the fact that $(\vec{G} - X, \gamma)$ does not contain a Γ -non-zero S -path. Hence, the \mathcal{T}_π -large block of $(\vec{G} - X, \gamma)$ is Γ -bipartite, as required.

Therefore, from now on, we assume that there are $4t$ disjoint Γ -non-zero S -paths $\mathcal{P} = \{P_1, \dots, P_{4t}\}$. We choose \mathcal{P} such that the number of edges lying in a path of \mathcal{P} but not in any tree $\eta(v_i)$ is as small as possible. By re-indexing, we may assume P_i has ends s_{2i-1} and s_{2i} . It is not difficult to see that $\bigcup_{i=8t+1}^{10t} \eta(v_i)$ does not intersect a path in \mathcal{P} by our choice of \mathcal{P} .

Let (\vec{G}', γ') be the minor of (\vec{G}, γ) obtained by contracting each $\eta(v_i)$ to a single vertex for $i \geq 8t+1$ and let $S' = \{s'_{8t+1}, \dots, s'_{10t}\}$ be the set of contracted vertices.

Again, by Lemma 19, there is a set $Y \subseteq V(G')$ such that $|Y| \leq 2t-2$ and $(\vec{G}' - Y, \gamma')$ does not contain a Γ -non-zero S' -path or (\vec{G}', γ') contains t disjoint Γ -non-zero S' -paths.

Suppose the first case holds. Since $|Y| \leq 2t-2$ and $|\mathcal{P}| = 4t$, we may assume (possibly after renaming) that Y is disjoint from $\eta(v_1), \eta(v_2), P_1, s'_{8t+1}$, and s'_{8t+2} . Note that P_1 may intersect $\eta(v_i)$ for some $i \in \{3, \dots, 8t\}$, and $E(P_1) \cap \eta(v_1 v_2) = \emptyset$, since η is Γ -bipartite. However, since $\eta(v_1) \cup \eta(v_2) \cup \eta(v_1 v_2) \cup P_1$ contains a Γ -non-zero cycle, there is also a Γ -non-zero s'_{8t+1} - s'_{8t+2} -path in (\vec{G}', γ') , which is a contradiction.

Therefore, we may assume that there are t disjoint Γ -non-zero S' -paths R'_1, \dots, R'_t in (\vec{G}', γ') . It is easy to see that we may lift these paths to t disjoint Γ -non-zero S -paths R_1, \dots, R_t in (\vec{G}, γ) such that for all $i \in [t]$, R_i joins $s_{8t+2i-1}$ and s_{8t+2i} , and R_i is disjoint from $\eta(v_j)$ for all $j \in \{8t+1, \dots, 10t\} \setminus \{8t+2i-1, 8t+2i\}$. By applying (1) to $\bigcup_{i=8t+1}^{10t} \eta(v_i) \cup R_i$, the proof of the lemma is complete. \square

We now extend Lemma 20 to (Γ_1, Γ_2) -group-labeled graphs.

Lemma 21. *For every $t \in \mathbb{N}$, there is an integer $T = T(t)$ such that for all groups Γ_1 and Γ_2 , every K_T -model π in a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph (\vec{G}, γ) either contains a (Γ_1, Γ_2) -non-zero K_t -model that is an enlargement of π , or a set $X \subseteq V(G)$ such that $|X| < 8R(t, 10t)$ and the \mathcal{T}_π -large block of $(\vec{G} - X, \gamma)$ is Γ_1 -bipartite or Γ_2 -bipartite.*

Proof. Let $r = R(t, 10t)$, $T = R(r, 10r)$, and π be a K_T -model in a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph (\vec{G}, γ) . By regarding π as a K_T -model in (\vec{G}, γ_1) and applying Lemma 20, there exists a Γ_1 -odd K_r -model π' in (\vec{G}, γ_1) , or a set of less than $8r$ vertices X_1 such that the \mathcal{T}_π -large block of $(\vec{G} - X_1, \gamma_1)$ is Γ_1 -bipartite.

We may assume the first case. By regarding π' as a K_r -model in (\vec{G}, γ_2) and applying Lemma 20, there exists a Γ_2 -odd K_t -model π'' in (\vec{G}, γ_2) that is an enlargement of π' , or a set of less than $8t$ vertices X_2 such that the $\mathcal{T}_{\pi'}$ -large block of $(\vec{G} - X_2, \gamma_2)$ is Γ_2 -bipartite.

Once again, we may assume that we have such a Γ_2 -odd K_t -model. Note that $|\pi''(uv)| = 1$ for every uv of π'' . For every vertex v of π'' , let v' be a vertex of π' such that $\pi'(v') \subseteq \pi''(v)$. By adding $\pi'(u'v')$ to $\pi''(uv)$ for every edge uv of π'' , we obtain a (Γ_1, Γ_2) -odd K_t -model in (\vec{G}, γ) . \square

6 A Flat Wall Theorem

In this section, we state our Flat Wall Theorem for $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs.

Let Γ_1 and Γ_2 be groups. Let (\vec{W}, γ) be a wall in a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph (\vec{G}, γ) . We say (\vec{W}, γ) is *facially* Γ_i -non-zero if every facial cycle of (\vec{W}, γ) is Γ_i -non-zero. If (\vec{W}, γ) is both facially Γ_1 -non-zero and facially Γ_2 -non-zero, then we say that (\vec{W}, γ) is *facially* (Γ_1, Γ_2) -non-zero.

Further suppose that (\vec{W}, γ) is a flat wall in (\vec{G}, γ) . Let (\vec{W}_0, γ) be a flat 1-contained subwall of (\vec{W}, γ) and let (A, B) be a separation certifying that W_0 is flat. Let N be the set of top nails of W_0 with respect to W .

Recall that for a family \mathcal{P} of disjoint N -paths, $I_{\mathcal{P}} \subseteq N$ is the minimal interval (under inclusion) containing all endpoints of paths in \mathcal{P} , and $I_{\mathcal{P}}^l$ and $I_{\mathcal{P}}^r$ are the minimal intervals containing all the left and right endpoints of \mathcal{P} , respectively. In the following we tacitly assume that N -paths are always traversed from their left endpoint to their right endpoint.

An N -linkage \mathcal{P} in (\vec{G}, γ) is *clean* with respect to (A, B) if

- (i) the paths in \mathcal{P} are internally disjoint from B ,
- (ii) \mathcal{P} is pure,
- (iii) every path in \mathcal{P} is Γ -non-zero, and
- (iv) if \mathcal{P} is crossing or nested, then $\gamma(P_1) = \gamma(P_2) \neq 0$ for all $P_1, P_2 \in \mathcal{P}$.

We say that \mathcal{P} is Γ_i -clean if it is clean in (\vec{G}, γ_i) . A pair of N -linkages $(\mathcal{P}, \mathcal{Q})$ is (Γ_1, Γ_2) -clean with respect to (A, B) if

- (i) \mathcal{P} and \mathcal{Q} are Γ_1 -clean and Γ_2 -clean, respectively,
- (ii) for all $P \in \mathcal{P}, Q \in \mathcal{Q}$, the paths P and Q are disjoint,
- (iii) $|\mathcal{P}| = |\mathcal{Q}|$,
- (iv) if $I_{\mathcal{P}} \cap I_{\mathcal{Q}} = \emptyset$, then $I_{\mathcal{P}} < I_{\mathcal{Q}}$ and otherwise if $I_{\mathcal{P}} \cap I_{\mathcal{Q}} \neq \emptyset$ then $I_{\mathcal{P}}^l < I_{\mathcal{Q}}^l < I_{\mathcal{P}}^r < I_{\mathcal{Q}}^r$ and neither \mathcal{P} nor \mathcal{Q} is in series.

The *size* of $(\mathcal{P}, \mathcal{Q})$ is $|\mathcal{P}|$. See Figure 3 for illustration. We can now state our refinement of the Flat Wall Theorem.

Theorem 22. *For every $t \in \mathbb{N}$, there exist integers $T(t)$ and $g(t)$ with the following property. Let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. If (\vec{G}, γ) contains a $T(t)$ -wall (\vec{W}, γ) , then one of the following statements holds.*

- (a) *There is a (Γ_1, Γ_2) -odd K_t -model π in (\vec{G}, γ) such that \mathcal{T}_{π} is a restriction of \mathcal{T}_W .*
- (b) *There is a set of vertices Z such that $|Z| \leq g(t)$ and there is a flat $100t$ -wall (\vec{W}_0, γ) in $(\vec{G} - Z, \gamma)$ with top nails N_0 and certifying separation (A_0, B_0) such that \mathcal{T}_{W_0} is a restriction of \mathcal{T}_W and after possibly shifting*
 - (b.i) *(\vec{W}_0, γ) is facially (Γ_1, Γ_2) -non-zero, or*
 - (b.ii) *for some $i \in [2]$, (\vec{B}_0, γ_i) is null-labeled, (\vec{W}_0, γ) is facially Γ_{3-i} -non-zero, and there is a Γ_i -clean N_0 -linkage \mathcal{P} of size t with respect to (A_0, B_0) , or*

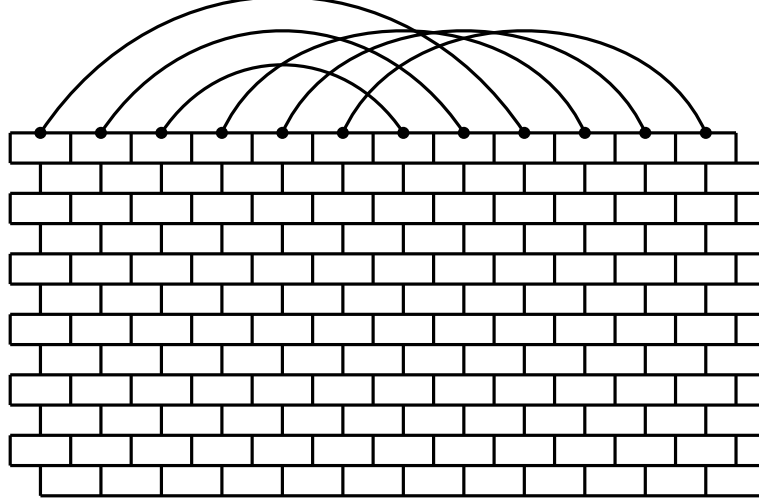


Figure 3: An example of a (Γ_1, Γ_2) -clean pair of linkages of size 3. The first three paths have the same (non-zero) group value in the first coordinate, and the other three paths have the same (non-zero) group value in the second coordinate.

(b.iii) (\vec{B}_0, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_0 -linkages of size t with respect to (A_0, B_0) .

(c) There is a set Z of vertices of (\vec{G}, γ) and some $i \in [2]$ such that $|Z| \leq g(t)$ and the \mathcal{T}_W -large block of

$$(\vec{G} - Z, \gamma)$$

is Γ_i -bipartite.

The Flat Wall Theorem for a group-labeled graph (\vec{G}, γ') with only one group Γ follows from Theorem 22 by considering the labeling $\gamma(e) = (\gamma'(e), \gamma'(e))$ for all edges $e \in E(G)$. After establishing some necessary lemmas in the next two sections, we will prove Theorem 22 in Section 9.

7 Cleaning Paths and Cycles

In this section, we prove some lemmas towards the proof of Theorem 22. Most of these lemmas are of the following form: given a family of paths or cycles, there is also a subfamily with certain nice properties.

Lemma 23. *Let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph that contains two disjoint cycles C_1 and C_2 and two disjoint C_1 - C_2 -paths P_1 and P_2 such that C_1 is Γ_1 -non-zero and C_2 is Γ_2 -non-zero. Then $C_1 \cup C_2 \cup P_1 \cup P_2$ contains a (Γ_1, Γ_2) -non-zero cycle.*

Proof. We may assume that $\gamma_2(C_1) = 0$ and $\gamma_1(C_2) = 0$; otherwise C_1 or C_2 is the desired (Γ_1, Γ_2) -non-zero cycle. Let C' be a cycle in $C_1 \cup C_2 \cup P_1 \cup P_2$ which contains $P_1 \cup P_2$. Again, if C' is (Γ_1, Γ_2) -non-zero, then we are done. Suppose v is

the first vertex of C' . Let C'_i be the cycle obtained by starting at v and following C' until intersecting $V(C_i)$, then following $C_i \setminus C'$ until intersecting $V(C')$, then following C' to v . Observe that $\gamma_{3-i}(C'_i) = \gamma_{3-i}(C')$ and $\gamma_i(C'_i) \neq \gamma_i(C')$. Hence if $\gamma_i(C') = 0$ and $\gamma_{3-i}(C') \neq 0$, then C'_i is a (Γ_1, Γ_2) -non-zero cycle. Finally, if $\gamma(C') = (0, 0)$, then $((C_1 \cup C_2) - C') \cup P_1 \cup P_2$ is a (Γ_1, Γ_2) -non-zero cycle. \square

Lemma 24. *Let Γ_1 and Γ_2 be groups, (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph, and C, C_1, C_2 be disjoint cycles of (\vec{G}, γ) . Let P_1, P'_1, P_2, P'_2 be pairwise disjoint paths such that P_i and P'_i are C - C_i -paths with $V(P_i) \cap V(C_{3-i}) = \emptyset = V(P'_i) \cap V(C_{3-i})$ for $i \in [2]$. Let p_1, p'_1, p_2, p'_2 be the ends of P_1, P'_1, P_2, P'_2 on C and assume that they occur in this cyclic order on C . Further assume that C_i is Γ_i -non-zero for every $i \in [2]$ and C_1 is Γ_2 -zero. Let I_1 be the subpath of C with ends p_1 and p'_2 and with $p_2 \notin V(I_1)$. Let I_2 be the subpath of C with ends p'_1 and p_2 and with $p_1 \notin V(I_2)$. Then there is a (Γ_1, Γ_2) -non-zero cycle D in $C_1 \cup C_2 \cup C \cup P_1 \cup P'_1 \cup P_2 \cup P'_2$ containing $I_1 \cup I_2$.*

Proof. We use the notation as in the statement. Moreover, let the endpoints of P_i and P'_i on C_i be denoted by q_i and q'_i , respectively. The vertices q_i and q'_i split C_i into q_i - q'_i -paths Q_i^a and Q_i^b . By shifting, we may assume that all edges in $I_1 \cup I_2 \cup P_1 \cup P'_1 \cup P_2 \cup P'_2$ are null-labeled.

Suppose $\gamma(Q_i^a) = (x_i^a, y_i^a)$ and $\gamma(Q_i^b) = (x_i^b, y_i^b)$. Since C_1 is Γ_2 -zero, we have $y_1^a = y_1^b := y_1$. Since C_2 is Γ_2 -non-zero, by symmetry, we may assume that $y_1 + y_2^a \neq 0$. Since C_1 is Γ_1 -non-zero, by symmetry, we may assume that $x_1^a + x_2^a \neq 0$. We are now done by letting D be

$$I_1 \cup I_2 \cup P_1 \cup P'_1 \cup P_2 \cup P'_2 \cup Q_1^a \cup Q_2^a. \quad \square$$

Lemma 25. *Let $t \in \mathbb{N}$, G be a graph, $L \subseteq V(G)$ be linearly ordered, and \mathcal{P} be an L -linkage. If $|\mathcal{P}| \geq t^3$, then \mathcal{P} contains a pure L -linkage \mathcal{P}' of size t .*

Proof. Dilworth's theorem [4] implies that for every pair of positive integers r, s , every collection of rs intervals contains r disjoint intervals or s pairwise intersecting intervals. For every path $P \in \mathcal{P}$, we associate an interval, namely, the smallest interval containing both endpoints of P . By Dilworth's theorem, we have t disjoint intervals or t^2 pairwise intersecting intervals. In the first case, \mathcal{P} contains t paths in series. So assume there are t^2 paths \mathcal{Q} such that their intervals contain a common point. We order these paths by their smaller endpoint. Their larger endpoints induce a permutation of their smaller endpoints. By the Erdős-Szekeres theorem [6], there is a monotone subsequence of order t . If this subsequence is increasing, then \mathcal{P} contains a crossing subfamily of size t , and if it is decreasing, then \mathcal{P} contains a nested subfamily of size t . \square

Lemma 26. *Let $t \in \mathbb{N}$, G be a graph, $L \subseteq V(G)$ be a linearly ordered set, and \mathcal{P}, \mathcal{Q} be two pure L -linkages of size $4t$ such that $\mathcal{P} \cup \mathcal{Q}$ is a L -linkage of size $8t$. Then, there exist two pure linkages $\mathcal{P}' \subseteq \mathcal{P}, \mathcal{Q}' \subseteq \mathcal{Q}$ of size t such that*

- $I_{\mathcal{P}'}$ and $I_{\mathcal{Q}'}$ are disjoint if \mathcal{P}, \mathcal{Q} are in series,
- $I_{\mathcal{P}'}$ is disjoint from $I_{\mathcal{Q}'}^\ell \cup I_{\mathcal{Q}'}^r$ if \mathcal{P} is in series and \mathcal{Q} is not in series,
- $I_{\mathcal{Q}'}$ is disjoint from $I_{\mathcal{P}'}^\ell \cup I_{\mathcal{P}'}^r$ if \mathcal{Q} is in series and \mathcal{P} is not in series, and
- $I_{\mathcal{P}'}^\ell, I_{\mathcal{P}'}^r, I_{\mathcal{Q}'}^\ell, I_{\mathcal{Q}'}^r$ are pairwise disjoint if neither \mathcal{P}, \mathcal{Q} is in series.

Proof. We order the paths of \mathcal{P} and \mathcal{Q} as $\{P_1, \dots, P_{4t}\}$ and $\{Q_1, \dots, Q_{4t}\}$ according to the order of their left endpoints. Let p_i^ℓ and p_i^r be the left and right endpoint of P_i and q_i^ℓ and q_i^r be the left and right endpoint of Q_i , respectively.

First suppose that \mathcal{P} is in series. If q_{3t}^ℓ is to the right of p_t^r , then $\mathcal{P}' = \{P_1, \dots, P_t\}$ and $\mathcal{Q}' = \{Q_{3t+1}, \dots, Q_{4t}\}$ are two families of paths with the desired properties. So we may assume that q_{3t}^ℓ is to the left of p_t^r . If \mathcal{Q} is also in series, then we may take $\mathcal{P}' = \{P_{t+1}, \dots, P_{2t}\}$ and $\mathcal{Q}' = \{Q_1, \dots, Q_t\}$. Thus \mathcal{Q} is either nested or crossing. If there are at least $3t$ right endpoints of \mathcal{Q} to the right of p_{2t}^r , then we may take $\mathcal{P}' = \{P_{t+1}, \dots, P_{2t}\}$ and $\mathcal{Q}' = \{Q_{t+1}, \dots, Q_{2t}\}$. Otherwise, we take \mathcal{Q}' to be a family of t paths of \mathcal{Q} all of whose right endpoints are to the left of p_{2t}^r , and $\mathcal{P}' = \{P_{2t+1}, \dots, P_{3t}\}$.

For the rest of the proof, we may assume that neither \mathcal{P} nor \mathcal{Q} are in series. For $i \in [4]$ define $X_i^\ell := [p_{1+(i-1)t}^\ell, p_{it}^\ell]$ and $X_i^r := [p_{1+(i-1)t}^r, p_{it}^r]$. Similarly, define $Y_i^\ell := [q_{1+(i-1)t}^\ell, q_{it}^\ell]$ and $Y_i^r := [q_{1+(i-1)t}^r, q_{it}^r]$. It suffices to find indices a and b such that $X_a^\ell \cup X_a^r$ is disjoint from $Y_b^\ell \cup Y_b^r$.

Let H be the graph with vertex set $\{X_i^\ell, X_i^r, Y_i^\ell, Y_i^r\}_{i \in [4]}$ and two vertices are adjacent if they intersect in L (considered as two intervals). By construction, H is an interval graph and bipartite. It is well known that interval graphs are chordal (that is, they do not contain induced cycles of length at least 4). Thus, H is a forest, because it is triangle-free. As H has 16 vertices, it has at most 15 edges.

If $X_a^\ell \cup X_a^r$ is not disjoint from $Y_b^\ell \cup Y_b^r$ for some $a, b \in [4]$, then $H_{a,b} = H[\{X_a^\ell, X_a^r, Y_b^\ell, Y_b^r\}]$ contains an edge. As $H_{a,b}$ is edge-disjoint from $H_{a',b'}$ for $(a,b) \neq (a',b')$ and there are 16 tuples (a,b) with $a, b \in [4]$, but only 15 edges in H , there exist (a,b) such that $H_{a,b}$ does not contain an edge. Therefore, $X_a^\ell \cup X_a^r$ is disjoint from $Y_b^\ell \cup Y_b^r$ as desired. \square

Lemma 27. *Let $t \in \mathbb{N}$, let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Let $S \subseteq V(G)$. If (\vec{G}, γ) contains $3t$ disjoint Γ_1 -non-zero S -paths and t disjoint Γ_2 -non-zero S -paths, then (\vec{G}, γ) contains $2t$ disjoint S -paths P_1, \dots, P_{2t} such that P_i is Γ_1 -non-zero for $i \leq t$ and Γ_2 -non-zero for $t < i \leq 2t$.*

Proof. Let $\mathcal{Q} = \{Q_1, \dots, Q_{3t}\}$ be disjoint Γ_1 -non-zero S -paths and let $\mathcal{R} = \{R_1, \dots, R_t\}$ be disjoint Γ_2 -non-zero S -paths such that the number of edges belonging to a path in \mathcal{R} but not to a path Q is as small as possible.

If the paths in \mathcal{R} intersect at most $2t$ paths of \mathcal{Q} , then we are done. Thus there is a path in \mathcal{Q} , say Q_1 , that intersects a path in \mathcal{R} , and the endpoints of Q_1 do not belong to a path in \mathcal{R} . Let q be an endpoint of Q_1 and let r be the first vertex of Q_1 after q contained in a path in \mathcal{R} , say R_1 . Note that the subpaths R_1r and rR_1 of R_1 both contain at least one edge that is not contained in a path in \mathcal{Q} . Moreover, since R_1 is a Γ_2 -non-zero path, at least one of the paths $qQ_1r \cup R_1r$ or $qQ_1r \cup rR_1$ is Γ_2 -non-zero, say $qQ_1r \cup R_1r$. Replacing R_1 by $qQ_1r \cup R_1r$ in \mathcal{R} contradicts the choice of \mathcal{R} . \square

Lemma 28. *Let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Let L be a linearly ordered subset of vertices of G and $t \in \mathbb{N}$. If (\vec{G}, γ) contains a Γ_1 -non-zero L -linkage of size $192t^3$ and a Γ_2 -non-zero L -linkage of size $64t^3$, then (\vec{G}, γ) contains two L -linkages \mathcal{P} and \mathcal{Q} of size t such that*

- (i) $\mathcal{P} \cup \mathcal{Q}$ is an L -linkage of size $2t$,

- (ii) the paths in \mathcal{P} and \mathcal{Q} are Γ_1 -non-zero and Γ_2 -non-zero, respectively,
- (iii) \mathcal{P}, \mathcal{Q} are both pure, and
- (iv)
 - $I_{\mathcal{P}}$ and $I_{\mathcal{Q}}$ are disjoint if \mathcal{P}, \mathcal{Q} are both in series,
 - $I_{\mathcal{P}}$ is disjoint from $I_{\mathcal{Q}}^{\ell}, I_{\mathcal{Q}}^r$ if \mathcal{P} is in series and \mathcal{Q} is not in series,
 - $I_{\mathcal{Q}}$ is disjoint from $I_{\mathcal{P}}^{\ell}, I_{\mathcal{P}}^r$ if \mathcal{Q} is in series and \mathcal{P} is not in series, and
 - $I_{\mathcal{P}}^{\ell}, I_{\mathcal{P}}^r, I_{\mathcal{Q}}^{\ell}, I_{\mathcal{Q}}^r$ are pairwise disjoint if neither \mathcal{P} nor \mathcal{Q} are in series.

Proof. First use Lemma 27 to obtain a Γ_1 -non-zero L -linkage \mathcal{P}'' of size $64t^3$ and a Γ_2 -non-zero L -linkage \mathcal{Q}'' of size $64t^3$ such that $\mathcal{P}'' \cup \mathcal{Q}''$ is an L -linkage of size $128t^3$. By Lemma 25, there are pure L -linkages $\mathcal{P}' \subseteq \mathcal{P}'', \mathcal{Q}' \subseteq \mathcal{Q}''$ each of size $4t$. Finally, Lemma 26 applied to $\mathcal{P}', \mathcal{Q}'$ completes the proof. \square

We now show how two large non-zero linkages attaching to a wall can be used to find a (Γ_1, Γ_2) -clean pair of linkages for a subwall. We first need an easy proposition on flat walls; we omit the proof.

Proposition 29. *Let G be a graph and $r \in \mathbb{N} \setminus \{1\}$. Let W be a flat r -wall in G and (A, B) be a certifying separation for W . Let X be the four corners of W and let C be the boundary cycle of W . Let H be a connected subgraph of G which is disjoint from W and such that $V(H)$ has a neighbor in each of the four components of $C - X$. Then $H \subseteq A$. Specifically, if W is a 1-contained subwall of a larger wall W' with boundary cycle C' , then $C' \subseteq A$.*

Lemma 30. *Let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Let $t \in \mathbb{N} \setminus \{1\}$ and W be a flat wall in G . Suppose W_1 is a 1-contained flat $10^6 t^6$ -subwall of W and let W_2 be a $10t$ -contained flat $10^2 t$ -subwall of W_1 . For $i \in [2]$, let (A_i, B_i) be a certifying separation for W_i . Let N_i be the set of top nails of W_i with respect to W . If (\vec{B}_1, γ) is null-labeled and there exist Γ_i -non-zero N_1 -linkages of size $10^5 t^6$ in $(\vec{A}_1 - (V(B_1) \setminus N_1), \gamma)$ for both $i \in [2]$, then there exists a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_2 -linkages for \vec{W}_2 of size t .*

Proof. First, we apply Lemma 28 in $(\vec{A}_1 - (V(B_1) \setminus N_1), \gamma)$ to obtain two N_1 -linkages \mathcal{P}_1 and \mathcal{Q}_1 each of size t^2 such that

- (i) $\mathcal{P}_1 \cup \mathcal{Q}_1$ is a linkage of size $2t^2$,
- (ii) the paths in \mathcal{P}_1 and \mathcal{Q}_1 are Γ_1 -non-zero and Γ_2 -non-zero, respectively,
- (iii) $\mathcal{P}_1, \mathcal{Q}_1$ are pure, and
- (iv)
 - $I_{\mathcal{P}_1}$ and $I_{\mathcal{Q}_1}$ are disjoint if $\mathcal{P}_1, \mathcal{Q}_1$ are in series,
 - $I_{\mathcal{P}_1}$ is disjoint from $I_{\mathcal{Q}_1}^{\ell}, I_{\mathcal{Q}_1}^r$ if \mathcal{P}_1 is in series and \mathcal{Q}_1 is not in series,
 - $I_{\mathcal{Q}_1}$ is disjoint from $I_{\mathcal{P}_1}^{\ell}, I_{\mathcal{P}_1}^r$ if \mathcal{Q}_1 is in series and \mathcal{P}_1 is not in series, and
 - $I_{\mathcal{P}_1}^{\ell}, I_{\mathcal{P}_1}^r, I_{\mathcal{Q}_1}^{\ell}, I_{\mathcal{Q}_1}^r$ are pairwise disjoint if neither \mathcal{P}_1 nor \mathcal{Q}_1 are in series.

We regard each path in $\mathcal{P}_1 \cup \mathcal{Q}_1$ as being traversed from its left endpoint to its right endpoint. Next, we define two new families \mathcal{P}_2 and \mathcal{Q}_2 of paths. If \mathcal{P}_1 is in series, then let \mathcal{P}_2 be an arbitrary subset of t paths in \mathcal{P}_1 . If \mathcal{P}_1 is nested or crossing and contains t paths with the same group value, then let \mathcal{P}_2 be a set of such paths. In the other cases, we order the paths in \mathcal{P} as P_1, \dots, P_{t^2} according to their left endpoints. For $i \in [t]$, choose $a_i > (i-1)t + 1$ such that $\gamma_1(P_{a_i}) \neq \gamma_1(P_{(i-1)t+1})$ with a_i minimum. Note that this implies that $a_i \leq it$. Let H be the topmost horizontal path of W_1 . For each $i \in [t]$, combine $P_{(i-1)t+1}$, P_{a_i} and the subpath of H between the right endpoints of $P_{(i-1)t+1}$ and P_{a_i} to a path and let \mathcal{P}_2 be the collection of these paths (observe that in this case the paths in \mathcal{P}_2 are in series). We do the same accordingly for \mathcal{Q}_1 to obtain \mathcal{Q}_2 . Note that $\mathcal{P}_2 \cup \mathcal{Q}_2$ is a collection of $2t$ disjoint paths and all paths in \mathcal{P}_2 are Γ_1 -non-zero and all paths in \mathcal{Q}_2 are Γ_2 -non-zero.

Since (\vec{B}_1, γ) is null-labeled, it is easy to see that $\mathcal{P}_2 \cup \mathcal{Q}_2$ can be extended to a (Γ_1, Γ_2) -clean pair of linkages for (\vec{W}_2, γ) by using paths in $W_1 - W_2$ and the fact that this subgraph is contained in A_2 by Proposition 29. \square

The following lemma is a simplified version of Lemma 30. The proof is the same as for Lemma 30.

Lemma 31. *Let Γ be a group and (\vec{G}, γ) be a Γ -labeled graph. Let $t \in \mathbb{N} \setminus \{1\}$ and W be a flat wall in G . Suppose W_1 is a 1-contained flat $10^6 t^6$ -subwall of W and let W_2 be a $10t$ -contained flat $10^2 t$ -subwall of W_1 . For $i \in [2]$, let (A_i, B_i) be a certifying separation for W_i . Let N_i be the set of top nails of W_i with respect to W . If (\vec{B}_1, γ) is null-labeled and there exist Γ -non-zero N_1 -linkages of size $10^5 t^6$ in $(\vec{A}_1 - (V(B_1) \setminus N_1), \gamma)$, then there exists a clean N_2 -linkage \mathcal{P} for \vec{W}_2 of size t .*

8 Wall Lemmas

In this section, we present several results concerning walls which we will use in the proof of Theorem 22. Recall, for a t -wall W , there exists a tangle \mathcal{T} of order $t+1$ associated with W . Explicitly, \mathcal{T} consists of all t -separations (A, B) of W such that B contains an entire horizontal path (or equivalently an entire vertical path) of W . If W is a minor of G (possibly $G = W$), then we let \mathcal{T}_W denote the tangle in G induced by \mathcal{T} .

Theorem 32 ([17] (7.5)). *For every positive integer t , there is an integer $T(t)$ with the following property. For every graph G and every tangle \mathcal{T} of order $T(t)$ in G , there is a t -wall W in G such that \mathcal{T}_W is a truncation of \mathcal{T} .*

Lemma 33. *Let s, t be positive integers and G be a graph. If W is a t -wall in G and W' is an s -subwall of W , then the tangle $\mathcal{T}_{W'}$ is a restriction of \mathcal{T}_W .*

Proof. Let (A, B) be an s -separation in G and assume that $(A, B) \in \mathcal{T}_{W'}$. Let P_0, \dots, P_s be the horizontal paths of W such that each contains as a subpath a horizontal path of W' . Since, $|V(A) \cap V(B)| \leq s$, there is some P_j such that $V(P_j)$ is disjoint from $V(A) \cap V(B)$. By the definition of $\mathcal{T}_{W'}$ it follows that $P_j \subseteq B$, and so $(A, B) \in \mathcal{T}_W$, as required. \square

Let W be an r -wall and let X be the branch vertices of the subdivision of the elementary r -wall. That is, X is the set of vertices of W corresponding to the vertices of the elementary r -wall before subdividing edges. Let $x, y \in X$ be contained in a common vertical path P and common brick B such that the subpath P' of P with ends x and y does not have any internal vertex in X and P' is not contained in the boundary cycle of W . Let Q be a W -path with ends x and y and let W' be the wall obtained from W by deleting the edges and internal vertices of P' and adding the path Q . In effect, we reroute the vertical path P through the path Q as opposed to the subpath P' . We say that W' is obtained from W via a *local rerouting*. We require the following easy proposition. We leave the proof to the reader.

Proposition 34. *Let W be a flat wall in a graph G . If W' is a local rerouting of W , then W' is also a flat wall in G and $\mathcal{T}_W = \mathcal{T}_{W'}$.*

We continue with two lemmas about walls which are the counterparts to Lemma 20 and 21 about K_t -minors.

Lemma 35. *Let $t \in \mathbb{N}$ and $t \geq 3$, let Γ be a group and (\vec{G}, γ) be a Γ -labeled graph. Let (\vec{W}, γ) be a flat $3t^2$ -wall in (\vec{G}, γ) . Then there exists a flat t -wall (\vec{W}_1, γ) with certifying separation (A_1, B_1) such that either*

- (i) *the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is Γ -bipartite, or*
- (ii) *(\vec{W}_1, γ) is facially Γ -non-zero.*

Moreover, \mathcal{T}_{W_1} is a restriction of \mathcal{T}_W and the boundary cycle of W is contained in A_1 .

Proof. Let (\vec{W}, γ) be a flat $3t^2$ -wall in (\vec{G}, γ) . The collection of all $((2t-1)i+1)$ -th horizontal paths and all $(2tj+1)$ -th vertical paths for $i, j \in \{0, \dots, t+1\}$ induces a 1-contained flat $(t+1)$ -subwall (\vec{W}_2, γ) of (\vec{W}, γ) . Note the interior of each brick $D_{i,j}$ of (\vec{W}_2, γ) contains a t -subwall $(\vec{W}_{i,j}, \gamma)$ of (\vec{W}, γ) for all $i, j \in [t+1]$. As a subwall of W , the wall $W_{i,j}$ is flat. Fix $(A_{i,j}, B_{i,j})$ to be a certifying separation for $W_{i,j}$ which minimizes $|V(B_{i,j})|$. This ensures that $B_{i,j}$ is connected and therefore $B_{i,j}$ is disjoint from $B_{i',j'}$ if either $i \neq i'$ or $j \neq j'$.

Assume that there exists $i, j \in [t+1]$ such that the block of $(\vec{B}_{i,j}, \gamma)$ containing $(\vec{W}_{i,j}, \gamma)$ is Γ -bipartite. As $W_{i,j}$ is a subwall of W , it follows that $\mathcal{T}_{W_{i,j}}$ is a restriction of \mathcal{T}_W by Lemma 33 and the boundary cycle of W is contained in $A_{i,j}$ by Proposition 29. Thus, the theorem holds.

We may therefore assume that for all $i, j \in [t+1]$, the block of $(\vec{B}_{i,j}, \gamma)$ containing $(\vec{W}_{i,j}, \gamma)$ also contains a Γ -non-zero cycle $C_{i,j}$. Let $H_{i,j}$ be the union of subpaths of the horizontal paths of W that are W_2 -paths which meet $W_{i,j}$. Let $i, j \in [t]$. Let $R_{i,j}$ be the subpath of the right vertical path of $D_{i,j}$ with endpoints in distinct horizontal paths of W_2 . By our choice of i and j , the path $R_{i,j}$ is not contained in the boundary cycle of W_2 . Observe that there are two disjoint $R_{i,j}$ - $C_{i,j}$ paths contained in $B_{i,j} \cup H_{i,j}$. Therefore, since $C_{i,j}$ is Γ -non-zero, there is a local rerouting of (\vec{W}_2, γ) along $R_{i,j}$ such that the (i, j) -th brick is Γ -non-zero. Note as well by the construction and Lemma 29 that after the local rerouting, the resulting wall is disjoint from the boundary cycle of W .

We sequentially perform these local reroutings along the right vertical path $R_{i,j}$ of each brick of (\vec{W}_2, γ) in lexicographic order $(1, 1), (2, 1), \dots, (t, t)$. By Lemma 33 and Proposition 34, the resulting $(t+1)$ -wall is flat and the induced tangle is a restriction of \mathcal{T}_W . Moreover, all the facial cycles except the last vertical and horizontal row are Γ -non-zero. To complete the proof, we delete the $(t+2)$ -th horizontal and $(t+2)$ -th vertical path to get a wall (\vec{W}_1, γ) which is facially non-zero and flat, satisfying (ii). Fix a certifying separation (A_1, B_1) of (\vec{W}_1, γ) . Let D_1 be the boundary cycle of W_1 . We conclude that the boundary cycle D of W is contained in A_1 by Proposition 29 applied to the component of $W - D_1$ containing D . This completes the proof. \square

We now extend Lemma 35 to the case when the graph is (Γ_1, Γ_2) -group-labeled. The proof follows the proof of Lemma 35, but the construction is slightly more complicated as we must use Lemma 24 to reroute the bricks to be (Γ_1, Γ_2) -non-zero.

Lemma 36. *Let Γ_1 and Γ_2 be groups, (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph, and $t \in \mathbb{N}$ with $t \geq 10$. Let (\vec{W}, γ) be a flat $9t^3$ -wall in (\vec{G}, γ) . Then there exists a flat t -wall (\vec{W}_1, γ) with certifying separation (A_1, B_1) that arises from a t -subwall of W via local rerouting such that one of the following statements holds.*

- (i) (\vec{W}_1, γ) is facially (Γ_1, Γ_2) -non-zero,
- (ii) for some $i \in [2]$, the wall (\vec{W}_1, γ) is facially Γ_i -non-zero and the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is Γ_{3-i} -bipartite, or
- (iii) the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is $(\Gamma_1 \oplus \Gamma_2)$ -bipartite.

Moreover, \mathcal{T}_{W_1} is a restriction of \mathcal{T}_W .

Proof. Let (\vec{W}, γ) be a flat $9t^3$ -wall in (\vec{G}, γ) . The collection of all $((8t^2-1)i+2)$ -th horizontal paths and $(8t^2j+1)$ -th vertical paths of (\vec{W}, γ) for all $i, j \in \{0, \dots, t+1\}$ induces a flat $(t+1)$ -subwall (\vec{W}_2, γ) of (\vec{W}, γ) .

For all $i, j \in [t+1]$, the interior of every brick $D_{i,j}$ of (\vec{W}_2, γ) contains two disjoint flat $3t^2$ -subwalls $(\vec{W}_{3,i,j}, \gamma)$ and $(\vec{W}_{4,i,j}, \gamma)$ such that they use the same set of vertical paths of W , and $W_{4,i,j}$ is above $W_{3,i,j}$. For $k \in \{3, 4\}$, fix a certifying separation $(A_{k,i,j}, B_{k,i,j})$ of $W_{k,i,j}$ which minimizes $|V(B_{k,i,j})|$.

Suppose that for some $i, j \in [t]$, the block of $(\vec{B}_{3,i,j}, \gamma)$ containing $(\vec{W}_{3,i,j}, \gamma)$ is Γ_1 -bipartite. By Lemma 35 applied to $(\vec{W}_{3,i,j}, \gamma_2)$ in the subgraph $(\vec{B}_{3,i,j}, \gamma)$, there exists a flat t -wall (\vec{W}_1, γ) with certifying separation (A_1, B_1) in $B_{3,i,j}$ such that the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is Γ_2 -bipartite or (\vec{W}_1, γ) is facially Γ_2 -non-zero. As the boundary cycle of $W_{3,i,j}$ is contained in A_1 , it follows that $(A_1 \cup A_{3,i,j}, B_1)$ is a certifying separation for W_1 in G . Moreover \mathcal{T}_{W_1} is a restriction of $\mathcal{T}_{3,i,j}$ and therefore also of \mathcal{T}_W .

Consider the two possible outcomes of Lemma 35. In the first case, the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is $(\Gamma_1 \oplus \Gamma_2)$ -bipartite, satisfying (iii). In the second case, (\vec{W}_1, γ) is facially Γ_2 -non-zero and the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is Γ_1 -bipartite, satisfying (ii). We conclude, by symmetry, that we may

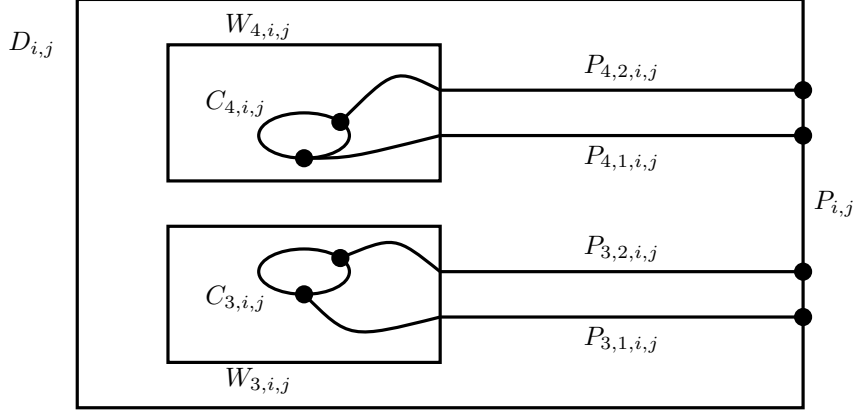


Figure 4: The figure displays the brick $D_{i,j}$ together with the walls $W_{1,i,j}, W_{4,i,j}$ in the proof of Lemma 36.

assume that for all $i, j \in [t]$ and $k \in \{3, 4\}$, the subgraph $(\vec{B}_{k,i,j}, \gamma)$ is neither Γ_1 -bipartite nor Γ_2 -bipartite.

Suppose next that for some $i, j \in [t]$ and $k \in \{3, 4\}$, the block of $(\vec{B}_{k,i,j}, \gamma)$ containing $(\vec{W}_{k,i,j}, \gamma)$ does not contain a cycle C such that C is

- Γ_1 -non-zero and Γ_2 -zero, or
- Γ_1 -zero and Γ_2 -non-zero.

By applying Lemma 35 to $W_{k,i,j}$ in $(\vec{B}_{k,i,j}, \gamma)$ with $\Gamma_1 \oplus \Gamma_2$ playing the role of Γ , we obtain a t -wall (\vec{W}_1, γ) with certifying separation (A_1, B_1) of $B_{k,i,j}$ such that either the block of (\vec{B}_1, γ) containing (\vec{W}_1, γ) is $(\Gamma_1 \oplus \Gamma_2)$ -bipartite, or (\vec{W}_1, γ) is facially $(\Gamma_1 \oplus \Gamma_2)$ -non-zero. As above, in each case, the boundary cycle of $W_{k,i,j}$ is contained in A_1 and so $(A_{k,i,j} \cup A_1, B_1)$ is a certifying separation that W_1 is flat as a subgraph of G . In the first case, we satisfy conclusion (iii). If the latter case holds, we note that (\vec{W}_1, γ) is actually facially (Γ_1, Γ_2) -non-zero, since it does not contain a cycle that is zero in exactly one coordinate, and hence we satisfy conclusion (i).

Therefore, we may assume that for all $i, j \in [t]$, there is a cycle $C_{3,i,j}$ in the block of $(\vec{B}_{3,i,j}, \gamma)$ containing $(\vec{W}_{3,i,j}, \gamma)$ that is Γ_k -zero and Γ_{3-k} -non-zero for some $k \in [2]$. Let $C_{4,i,j}$ be a cycle in the block of $(\vec{B}_{4,i,j}, \gamma)$ containing $(\vec{W}_{4,i,j}, \gamma)$ that is Γ_k -non-zero.

Let $P_{i,j}$ be the subpath of the right vertical path of the brick $D_{i,j}$ with endpoints in distinct horizontal paths of W_3 . It is easy to see that there exist four disjoint paths $P_{3,1,i,j}, P_{3,2,i,j}, P_{4,1,i,j}, P_{4,2,i,j}$ such that the following hold. See Figure 4 for illustration.

- $P_{3,1,i,j}$ and $P_{3,2,i,j}$ join $C_{3,i,j}$ and $P_{i,j}$,
- $P_{4,1,i,j}$ and $P_{4,2,i,j}$ join $C_{4,i,j}$ and $P_{i,j}$, and
- the paths $P_{3,1,i,j}, P_{3,2,i,j}, P_{4,1,i,j}, P_{4,2,i,j}$ end in this order on $P_{i,j}$.

Next we reroute similarly as in the proof of Lemma 35. We start by applying Lemma 24 to the outer cycle of $B_{1,1}$ and $P_{3,1,1,1}$, $P_{3,2,1,1}$, $P_{4,1,1,1}$, $P_{4,2,1,1}$, $C_{3,1,1}$, $C_{4,1,1}$. This gives a local rerouting of $P_{1,1}$ such that the brick $D_{1,1}$ has been modified to a brick that is a (Γ_1, Γ_2) -non-zero.³ We then continue iteratively in lexicographic order to find a local rerouting for all bricks $D_{i,j}$ for $i, j \in [t]$. Note that once a brick is adjusted, it is never modified again. At the end, we obtain a flat $(t+1)$ -subwall where every facial cycle is non-zero except possibly the last vertical column or horizontal row. After deleting the $(t+2)$ -horizontal and $(t+2)$ -vertical path, we obtain a flat t -wall (\vec{W}_1, γ) of (\vec{G}, γ) that is facially (Γ_1, Γ_2) -non-zero. By construction and Proposition 34, \mathcal{T}_{W_1} is a restriction of \mathcal{T}_W as required. \square

Lemma 37. *Let $t, r \in \mathbb{N}$ such that $r \geq 3t$ and $t \geq 2$. Let Γ be a group, and let (\vec{G}, γ) be a Γ -labeled graph. Let W be a flat wall in G with certifying separation (A, B) such that (\vec{B}, γ) is null-labeled and W $2t$ -contains an r -subwall W_1 . Let N_1 be the top nails of W_1 with respect to W . Let (A_1, B_1) be a certifying separation for W_1 . Let X be a set of vertices of G meeting all Γ -non-zero N_1 -paths in $(\vec{G} - (V(B) \setminus N_1), \gamma)$. If $|X| \leq 2t - 2$, then the \mathcal{T}_W -large block of $(\vec{G} - X, \gamma)$ is Γ -bipartite.*

Proof. Let X be a set of at most $2t - 2$ vertices meeting every Γ -non-zero N_1 -path in $(\vec{G} - (V(B) \setminus N_1), \gamma)$. Let (\vec{U}, γ) be the \mathcal{T}_W -large block of $(\vec{G} - X, \gamma)$. For a contradiction, suppose (\vec{U}, γ) contains a Γ -non-zero cycle C .

Because $|X| \leq 2t - 2$ and W_1 is $2t$ -contained in W , there is a vertical path of W to the left of W_1 and one to the right of W_1 and also a horizontal path of W below W_1 and one above W_1 , all having an empty intersection with X . Clearly, the union of these four paths contains a cycle D that is contained in U . Moreover, since there are at least two vertical paths of W containing a vertex of N_1 and disjoint from X , there are two disjoint paths in W linking D and two vertices of N_1 . Let the union of these two paths and D be denoted by \vec{D} . Since \vec{D} is contained in (\vec{W}, γ) , all edges of \vec{D} are null-labeled.

Consider the subgraph B of G . By assumption, there exists an $(A \cap B)$ -reduction B' of B which embeds in a closed disk Δ . The graph B' is obtained from B by repeatedly performing elementary $V(A \cap B)$ -reductions. Thus the graph B' is a subgraph of B with some additional edges added through the elementary reductions. The cycle D corresponds to a cycle D' of B' which bounds a disk Δ' in Δ . We want to consider the subgraph H of B which can be thought of as laying ‘inside’ the disk Δ' . We begin by defining H' to be the subgraph of $B' \cap B$ embedded in the disc Δ' . Define H to be the union of the subgraph H' along with B^* for any separation (A^*, B^*) of G of order at most 3 where $V(A^* \cap B^*)$ is contained in $V(H')$. Observe that the cycle C is not a subgraph of H because H is a subgraph of B and therefore Γ -bipartite. Observe as well that W_1 is a subgraph of H .

Suppose that C is disjoint from H . Because U is a block, there are two disjoint paths in U linking D and C . Combining these paths with suitable portions of \vec{D} and C yields a non-zero N_1 -path in $(\vec{G} - (V(B_1) \setminus N_1), \gamma)$ that avoids all vertices in X , which is a contradiction. Therefore, we may assume

³ Note that in the application of Lemma 24, we may possibly have to interchange the role of Γ_1 and Γ_2 in the statement.

that D decomposes C into a family of internally disjoint paths P_1, \dots, P_{2n} that alternate between being contained in H and being disjoint from H (except for their endpoints). All the paths contained in H are Γ -zero since (\vec{B}, γ) is null-labeled. Therefore, there is a path $P \in \mathcal{P}$ that is disjoint from H (other than its endpoints) and Γ -non-zero. Combining P with a suitable portion of \vec{D} , again leads to a Γ -non-zero N_1 -path in $(\vec{G} - (V(B_1) \setminus N_1), \gamma)$ that avoids all vertices in X , which is a contradiction. \square

9 Proof of our Flat Wall Theorem

We are now ready to prove Theorem 22, which we restate for the reader's convenience.

Theorem 22. *For every $t \in \mathbb{N}$, there exist integers $T(t)$ and $g(t)$ with the following property. Let Γ_1 and Γ_2 be groups and (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. If (\vec{G}, γ) contains a $T(t)$ -wall (\vec{W}, γ) , then one of the following statements holds.*

- (a) *There is a (Γ_1, Γ_2) -odd K_t -model π in (\vec{G}, γ) such that \mathcal{T}_π is a restriction of \mathcal{T}_W .*
- (b) *There is a set of vertices Z such that $|Z| \leq g(t)$ and there is a flat $100t$ -wall (\vec{W}_0, γ) in $(\vec{G} - Z, \gamma)$ with top nails N_0 and certifying separation (A_0, B_0) such that \mathcal{T}_{W_0} is a restriction of \mathcal{T}_W and after possibly shifting*
 - (b.i) *(\vec{W}_0, γ) is facially (Γ_1, Γ_2) -non-zero, or*
 - (b.ii) *for some $i \in [2]$, (\vec{B}_0, γ_i) is null-labeled, (\vec{W}_0, γ) is facially Γ_{3-i} -non-zero, and there is a Γ_i -clean N_0 -linkage \mathcal{P} of size t with respect to (A_0, B_0) , or*
 - (b.iii) *(\vec{B}_0, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_0 -linkages of size t with respect to (A_0, B_0) .*
- (c) *There is a set Z of vertices of (\vec{G}, γ) and some $i \in [2]$ such that $|Z| \leq g(t)$ and the \mathcal{T}_W -large block of*

$$(\vec{G} - Z, \gamma)$$
is Γ_i -bipartite.

Proof. Let \mathcal{T}_W be the tangle in G induced by W . We first suppose that there is a $T(t)$ -clique minor K in G (where $T(t)$ is the function from Lemma 21) such that \mathcal{T}_K is a restriction of \mathcal{T}_W . By Lemma 21, there is a (Γ_1, Γ_2) -odd K_t -model π in (\vec{G}, γ) such that \mathcal{T}_π is a restriction of \mathcal{T}_W or there is a set Z of vertices (with $|Z|$ bounded by a function of t) and an index $i \in [2]$ such that the \mathcal{T}_K -large block of $(\vec{G} - Z, \gamma)$ is Γ_i -bipartite. Since \mathcal{T}_K is a restriction of \mathcal{T}_W , this block is also \mathcal{T}_W -large, so we are done.

Therefore, we may assume that \mathcal{T}_W does not control a large clique-minor. By the Flat Wall Theorem (Theorem 16), there is a set of vertices Z_1 such that $|Z_1|$ is bounded in terms of t and W contains a $10^{22}t^{18}$ -subwall W_1 which is flat in $G_1 = G - Z_1$.

By Lemma 36, there exists a flat 10^7t^6 -wall W_2 with certifying separation (A_2, B_2) such that

- (i) (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, or
- (ii) (\vec{W}_2, γ) is facially Γ_i -non-zero and (\vec{B}_2, γ) is Γ_{3-i} -bipartite for some $i \in [2]$,
or
- (iii) (\vec{B}_2, γ) is $(\Gamma_1 \oplus \Gamma_2)$ -bipartite.

Moreover, \mathcal{T}_{W_2} is a restriction of \mathcal{T}_{W_1} .

If (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, then we are done. Otherwise, let W_3 be a $10^6 t^6$ -subwall of W_2 that is 1-contained in W_2 . Let N_3 be the top nails of W_3 with respect to W_2 . The wall W_3 is flat since it is a subwall of W_2 . Fix a certifying separation (A_3, B_3) for W_3 with $|V(B_3)|$ minimum.

Let H be a component of A_2 containing a vertex of $A_2 \cap B_2$. The union of H along with the component of $W_2 - V(W_3)$ containing the boundary cycle of W_2 is a connected subgraph of G which is disjoint from W_3 and has as neighbors every nail and corner of W_3 in the boundary cycle of W_3 . It follows from Proposition 29 that H is a subgraph of A_3 . We conclude that B_3 is a subgraph of B_2 .

Suppose (ii) holds; that is, (\vec{W}_2, γ) is facially Γ_i -non-zero and (\vec{B}_2, γ) is Γ_{3-i} -bipartite for some $i \in [2]$. We perform shifts so that all edges in $(\vec{B}_2, \gamma_{3-i})$ are null-labeled. Let W_4 be a $10t$ -contained $100t$ -subwall of W_3 (such that every brick of W_4 is a brick of W_3) with certifying separation (A_4, B_4) such that $|V(B_4)|$ is minimal. Let N_4 be the top nails of W_4 with respect to W_3 .

If there is a set of vertices Z_2 of size at most $2 \cdot 10^5 t^6$ meeting all Γ_{3-i} -non-zero N_3 -paths in $(\vec{G}_1 - (V(B_3) \setminus N_3), \gamma)$, then by Lemma 37, the \mathcal{T}_{W_3} -large block of $(\vec{G}_1 - Z_2, \gamma)$ is Γ_{3-i} -bipartite. Therefore, $Z = Z_1 \cup Z_2$ satisfies the third outcome of the theorem.

Thus, we may suppose that such a set Z_2 does not exist. Applying Lemma 19 gives $10^5 t^6$ disjoint Γ_{3-i} -non-zero N_3 -paths in $(\vec{G}_1 - (V(B_3) \setminus N_3), \gamma)$. Applying Lemma 31 yields a Γ_{3-i} -clean W_4 -linkage of size t , as required.

Therefore, we may assume that (iii) holds; that is, (\vec{B}_2, γ) is $(\Gamma_1 \oplus \Gamma_2)$ -bipartite. We perform shifts so that all edges in (\vec{B}_2, γ) are null-labeled. As B_3 is a subgraph of B_2 , all the edges in (\vec{B}_3, γ) are null-labeled. By the previous argument, we may assume that there are $10^5 t^6$ disjoint Γ_i -non-zero N_3 -paths in $(\vec{G}_1 - (V(B_3) \setminus N_3), \gamma)$ for both $i \in [2]$. Applying Lemma 30 yields a (Γ_1, Γ_2) -clean pair of N_4 -linkages of size t , which completes the proof of the theorem. \square

10 Deriving the Erdős-Pósa results

We finish our paper by deriving Theorem 1 and 2 from Theorem 22. We also give some additional applications. For the reader's convenience we restate both theorems.

Theorem 1. *For every integer k , there exists an integer $f(k)$ with the following property. Let Γ_1 and Γ_2 be groups and let (\vec{G}, γ) be a $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Then, (\vec{G}, γ) contains k (Γ_1, Γ_2) -non-zero cycles such that each vertex of (\vec{G}, γ) is in at most two of these cycles, or there exists a set of at most $f(k)$ vertices of G such that $(\vec{G} - X, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle.*

Proof. Let f be a fast growing function to be specified later. Assume for a contradiction that $((\vec{G}, \gamma), k)$ is a minimal counterexample showing that f is not a half-integral Erdős-Pósa function for the set of $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. By Lemma 13 and Theorem 32, there is a tangle \mathcal{T} of order T_a and a T_b -wall W_1 in G such that \mathcal{T}_{W_1} is a restriction of \mathcal{T} . We specify T_b later and choose T_a sufficiently large in terms of T_b , as required by Theorem 32.

Next we apply Theorem 22 to (\vec{W}_1, γ) and choose T_b sufficiently large so that we can apply Theorem 22 with $t = 6k$. If there is a (Γ_1, Γ_2) -odd K_{6k} -minor in (\vec{G}, γ) , then, by Lemma 23 (applied k times to two disjoint cycles given by two disjoint triangles of the K_{6k} -model), there are k disjoint (Γ_1, Γ_2) -non-zero cycles in (\vec{G}, γ) , which is a contradiction to our assumption.

If there is a set $Z \subseteq V(G)$ such that $|Z| \leq g(6k)$ (the function g of Theorem 22) and the \mathcal{T} -large block of $(\vec{G} - Z, \gamma)$ is Γ_i -bipartite for some $i \in [2]$, then $(\vec{G} - Z, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle, which is a contradiction to our assumption.

Therefore, by Theorem 22, we may assume that there exists a set $Z \subseteq V(G)$ such that $|Z| \leq g(6k)$ and $G - Z$ contains a flat $600k$ -wall W_2 with certifying separation (A_2, B_2) and top nails N_2 such that \mathcal{T}_{W_2} is a restriction of \mathcal{T}_{W_1} and after possibly shifting

- (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, or
- for some $i \in [2]$, the wall (\vec{B}_2, γ_i) is null-labeled, (\vec{W}_2, γ) is facially Γ_{3-i} -non-zero, and there is a Γ_i -clean N_2 -linkage \mathcal{P} with respect to (A_2, B_2) of size k , or
- (\vec{B}_2, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_2 -linkages of size $2k$ with respect to (A_2, B_2) .

If (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, then clearly (\vec{G}, γ) contains k disjoint (Γ_1, Γ_2) -non-zero cycles, which is a contradiction.

Suppose next, by symmetry, that (\vec{B}_2, γ_1) is null-labeled, (\vec{W}_2, γ) is facially Γ_2 -non-zero, and there is a Γ_1 -clean N_2 -linkage $\mathcal{P} = \{P_1, \dots, P_k\}$. The endpoints of the paths in \mathcal{P} inherit a linear ordering from N_2 . We extend each P_i to a path P'_i as follows. First extend each P_i from each of its ends to the next smaller branch vertex (or corner) of W_2 . Next, extend each of these new paths via the vertical paths of W_2 so that P'_i crosses exactly $10i$ horizontal paths and for each horizontal path H which intersects P'_i , the graph $H \cap P'_i$ has two components. Let $a(i)$ be the integer such that the endpoints of P'_i lie on the horizontal path $P_{a(i)}(h)$ of W_2 . Let H_i be the subpath of $P_{a(i)}(h)$ that joins the ends of P'_i , and let D_i be a brick of W_2 intersecting H_i in at least one edge. Since D_i is a Γ_2 -non-zero cycle, either $P'_i \cup H_i$ or $(P'_i \cup H_i) \Delta D_i$ is a (Γ_1, Γ_2) -non-zero cycle where Δ denotes the symmetric difference. In total, we have constructed k (Γ_1, Γ_2) -non-zero cycles, and every vertex of G is contained in at most two cycles, which is a contradiction.

Finally, suppose (\vec{B}_2, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_2 -linkages of size $2k$. If \mathcal{P} or \mathcal{Q} contain k paths that are (Γ_1, Γ_2) -non-zero, then using these k paths and the null-labeled wall W_2 in the same construction as the previous paragraph yields a set of k (Γ_1, Γ_2) -non-zero cycles

such that every vertex is in at most two of the cycles, a contradiction. Hence, we may assume that there is a (Γ_1, Γ_2) -clean pair of N_2 -linkages $(\mathcal{P}', \mathcal{Q}')$ of size k such that every path in \mathcal{P}' is Γ_1 -non-zero and Γ_2 -zero and every path in \mathcal{Q}' is Γ_1 -zero and Γ_2 -non-zero. Extending the paths in $\mathcal{P}' \cup \mathcal{Q}'$ similarly as in the previous case yields k (Γ_1, Γ_2) -non-zero cycles each containing exactly one path of \mathcal{P} , one path of \mathcal{Q} , and the subpaths of two distinct horizontal paths of W_2 . The cycles can be constructed such that every vertex of G is contained in at most two of these cycles, which is the final contradiction. \square

Theorem 2. *For every integer k , there exists an integer $f(k)$ with the following property. Let Γ_1 and Γ_2 be groups and let (\vec{G}, γ) be a robust $(\Gamma_1 \oplus \Gamma_2)$ -labeled graph. Then, (\vec{G}, γ) contains k disjoint (Γ_1, Γ_2) -non-zero cycles or there exists a set of at most $f(k)$ vertices of (\vec{G}, γ) such that $(\vec{G} - X, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle.*

Proof. The proof is similar to the proof of Theorem 1. Let f be a fast growing function to be specified later. Assume for a contradiction that $((\vec{G}, \gamma), k)$ is a minimal counterexample showing that f is not an Erdős-Pósa function for the set of robust $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs. By Lemma 13 and Theorem 32, there is a tangle \mathcal{T} of order T_a and a T_b -wall W_1 in G such that \mathcal{T}_{W_1} is a restriction of \mathcal{T} . We specify T_b later and choose T_a sufficiently large in terms of T_b , as required by Theorem 32.

Next we apply Theorem 22 to (\vec{W}_1, γ) and choose T_b sufficiently large so that we can apply Theorem 22 with $t = 6k$. If there is a (Γ_1, Γ_2) -odd K_{6k} -minor in (\vec{G}, γ) , then by Lemma 23, there are k disjoint (Γ_1, Γ_2) -non-zero cycles in (\vec{G}, γ) , which is a contradiction to our assumption.

If there is a set $Z \subseteq V(G)$ such that $|Z| \leq g(6k)$ (the function g of Theorem 22) and the \mathcal{T} -large block of $(\vec{G} - Z, \gamma)$ is Γ_i -bipartite for some $i \in [2]$, then $(\vec{G} - Z, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle, which is a contradiction to our assumption.

Therefore, by Theorem 22, we may assume that there exists a set $Z \subseteq V(G)$ such that $|Z| \leq g(6k)$ and $G - Z$ contains a flat $600k$ -wall W_2 with certifying separation (A_2, B_2) and top nails N_2 such that \mathcal{T}_{W_2} is a restriction of \mathcal{T}_{W_1} and after possibly shifting

- (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, or
- for some $i \in [2]$, (\vec{B}_2, γ_i) is null-labeled, (\vec{W}_2, γ) is facially Γ_{3-i} -non-zero, and there is a Γ_i -clean N_2 -linkage \mathcal{P} of size k with respect to (A_2, B_2) , or
- (\vec{B}_2, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair $(\mathcal{P}, \mathcal{Q})$ of N_2 -linkages with respect to (A_2, B_2) of size $2k$.

If (\vec{W}_2, γ) is facially (Γ_1, Γ_2) -non-zero, then clearly (\vec{G}, γ) contains k disjoint (Γ_1, Γ_2) -non-zero cycles, which is a contradiction.

Suppose next, by symmetry, that (\vec{B}_2, γ_1) is null-labeled, (\vec{W}_2, γ) is facially Γ_2 -non-zero, and there is a Γ_1 -clean N_2 -linkage \mathcal{P} of size k . If \mathcal{P} is in series or nested, then it is easy to see that we can extend k paths in \mathcal{P} to a set of k disjoint (Γ_1, Γ_2) -non-zero cycles, which is a contradiction. Observe that \mathcal{P} cannot be crossing as (\vec{G}, γ) is robust. To see this, observe that if \mathcal{P} is crossing,

then for all paths $P_1, P_2 \in \mathcal{P}$, we have $\gamma_1(P_1) = \gamma_1(P_2)$. Therefore, it is easy to construct two cycles C_1, C_2 such that C_i consists of P_i and a (null-labeled) path in W_2 for each $i \in [2]$. These two cycles contradict the fact (\vec{G}, γ) is robust.

Finally, suppose (\vec{B}_2, γ) is null-labeled and there is a (Γ_1, Γ_2) -clean pair of N_2 -linkages $(\mathcal{P}, \mathcal{Q})$ of size $2k$. If \mathcal{P} (or \mathcal{Q}) contains k paths that are (Γ_1, Γ_2) -non-zero, then we can extend these paths to k disjoint (Γ_1, Γ_2) -non-zero cycles. Note that in order to do so, we need that \mathcal{P} is not crossing, which holds given that (\vec{G}, γ) is robust.

Hence we may assume that there is a (Γ_1, Γ_2) -clean pair of N_2 -linkages $(\mathcal{P}', \mathcal{Q}')$ of size k such that each path in \mathcal{P}' is Γ_1 -non-zero and Γ_2 -zero, and each path in \mathcal{Q}' is Γ_1 -zero and Γ_2 -non-zero. If \mathcal{P}' and \mathcal{Q}' are of the same type, then there exist k disjoint (Γ_1, Γ_2) -non-zero cycles, where each cycle uses exactly one path from each of \mathcal{P}' and \mathcal{Q}' . We conclude that, by symmetry, \mathcal{P}' is in series and \mathcal{Q}' is nested. However, this is a contradiction, because \mathcal{Q}' cannot be nested as (\vec{G}, γ) is robust. \square

Remark. In the proof of Theorem 2, the robustness of (G, γ) is only used in three places. The first is if the clean N_2 -linkage \mathcal{P} is a crossing linkage. The second is if in the (Γ_1, Γ_2) -clean pair of N_2 -linkages $(\mathcal{P}, \mathcal{Q})$, one of \mathcal{P} or \mathcal{Q} is crossing and contains k paths that are (Γ_1, Γ_2) -non-zero. The third is if the (Γ_1, Γ_2) -clean pair of W -linkages $(\mathcal{P}', \mathcal{Q}')$ are of different types. If we allow these as separate outcomes, then we obtain the following theorem for the full Erdős-Pósa-property for (Γ_1, Γ_2) -non-zero cycles.

Theorem 38. *For every integer k , there exists an integer $f(k)$ with the following property. For all groups Γ_1 and Γ_2 , and all $(\Gamma_1 \oplus \Gamma_2)$ -labeled graphs (\vec{G}, γ) , at least one of the following holds.*

- (i) (\vec{G}, γ) contains k disjoint (Γ_1, Γ_2) -non-zero cycles.
- (ii) There exists a set of at most $f(k)$ vertices of G such that $(\vec{G} - X, \gamma)$ does not contain any (Γ_1, Γ_2) -non-zero cycle.
- (\vec{G}, γ) contains a $600k$ -wall (\vec{W}, γ) with top nails N satisfying one of the following.
 - (iii) There exists a crossing N -linkage \mathcal{P} of size k which is internally disjoint from W such that after possibly shifting there exists an index $i \in [2]$ such that (\vec{W}, γ_i) is null-labeled, (\vec{W}, γ) is facially Γ_{3-i} -non-zero, and for all $P_1, P_2 \in \mathcal{P}$, we have $\gamma_i(P_1) = \gamma_i(P_2) \neq 0$, or
 - (iv) there exists a crossing N -linkage \mathcal{P} of size k which is internally disjoint from W such that after possibly shifting (\vec{W}, γ) is null-labeled, and each path in \mathcal{P} is (Γ_1, Γ_2) -non-zero, or
 - (v) there exists a pair of N -linkages $(\mathcal{P}, \mathcal{Q})$ of size k such that \mathcal{P} and \mathcal{Q} are of different type and after possibly shifting
 - $(\mathcal{P}, \mathcal{Q})$ is a (Γ_1, Γ_2) -clean N -linkage,
 - (\vec{W}, γ) is null-labeled, and
 - P is Γ_1 -zero and Γ_2 -non-zero for all $P \in \mathcal{P}$, and

- Q is Γ_1 -non-zero and Γ_2 -zero for all $Q \in \mathcal{Q}$.

Let G be a graph and $S \subseteq V(G)$. Let \vec{G} be an arbitrary orientation of G and let e_1, \dots, e_m be an enumeration of $E(\vec{G})$. Define $\gamma : E(\vec{G}) \rightarrow \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}$ by $\gamma(e_i) = (1, 2^i)$, if e_i has at least one end in S , and $\gamma(e_i) = (1, 0)$, otherwise. By applying Theorem 38 to (\vec{G}, γ) , we obtain the following theorem for odd S -cycles.

Theorem 39. *For every positive integer k , there exist an integer $f(k)$ with the following property. For all graphs G and all $S \subseteq V(G)$ at least one of the following holds.*

- (i) G contains k disjoint odd S -cycles,
- (ii) there exists a set $X \subseteq V(G)$ such that $G - X$ does not contain an odd S -cycle and $|X| \leq f(k)$, or
- (iii) G contains a bipartite $600k$ -wall W with top nails N such that every face of W contains a vertex of S , and there is a crossing N -linkage \mathcal{P} of size k which is internally disjoint from W such that $W \cup P$ is not bipartite for all $P \in \mathcal{P}$, or
- (iv) G contains a bipartite $600k$ -wall W with top nails N such that $V(W) \cap S = \emptyset$, and there is a crossing N -linkage \mathcal{P} of size k which is internally disjoint from W such that $W \cup P$ is not bipartite and $V(P) \cap S \neq \emptyset$ for all $P \in \mathcal{P}$, or
- (v) G contains a bipartite $600k$ -wall W with top nails N such that $V(W) \cap S = \emptyset$, and there is a pair of pure N -linkages \mathcal{P} and \mathcal{Q} each of size k such that
 - $\mathcal{P} \cup \mathcal{Q}$ is a linkage of size $2k$,
 - \mathcal{P} is in series and \mathcal{Q} is either crossing or nested,
 - $V(P) \cap S \neq \emptyset$ and $P \cup W$ is bipartite for all $P \in \mathcal{P}$,
 - $V(Q) \cap S = \emptyset$ and $Q \cup W$ is not bipartite for all $Q \in \mathcal{Q}$, and
 - $I_{\mathcal{P}} \cap I_{\mathcal{Q}} = \emptyset$.

Remark. If $S = V(G)$, the last two outcomes of Theorem 39 obviously cannot occur. Therefore, we obtain an independent proof of Reed's Escher-wall Theorem (Theorem 18).

As a last application, we present the canonical set of obstructions for cycles not homologous to zero. Let Σ be a surface, $\mathcal{H}(\Sigma)$ be the (first) homology group of Σ , and G be a graph embedded in Σ . Recall the construction of a $(\mathcal{H}(\Sigma) \oplus \mathcal{H}(\Sigma))$ -labeled graph (\vec{G}, γ) from G as follows. Let \vec{G} be an arbitrary orientation of G . Fix a spanning forest F of G , and define $\gamma(e) = (0, 0)$ for all $e \in \vec{F}$. For $e \notin \vec{F}$, define $\gamma(e)$ to be (α, α) , where α is the homology class of the unique cycle contained in $e \cup F$. If we apply Theorem 38 to (G, γ) , then outcome (iii) and (v) cannot occur since $\gamma_1(C) = \gamma_2(C)$ for all cycles C in (G, γ) . Therefore, we immediately obtain the following theorem.

Theorem 40. *For every integer k there exists an integer $f(k)$ with the following property. For every surface Σ and every graph G embedded in Σ at least one of the following holds.*

- (i) G contains k disjoint cycles not homologous to zero,
- (ii) there is a set of at most $f(k)$ vertices of G such that all cycles of $G - X$ are homologous to zero, or
- (iii) G contains a $600k$ -wall W with top nails N and a crossing N -linkage \mathcal{P} of size k which is internally disjoint from W such that each path in \mathcal{P} goes through the same crosscap of Σ .

As an immediate corollary of Theorem 40, we obtain the (full) Erdős-Pósa property for cycles not homologous to zero on an orientable surface.

Corollary 41. *For every integer k there exists an integer $f(k)$ with the following property. For every orientable surface Σ and every graph G embedded on Σ , either G contains k disjoint cycles not homologous to zero, or there is a set of at most $f(k)$ vertices of G such that all cycles of $G - X$ are homologous to zero.*

Remark. Fix $\ell \in \mathbb{N}$. We call a cycle in a (Γ_1, Γ_2) -group-labeled graph (\vec{G}, γ) *long* if it has length at least ℓ . Note that the proofs of Theorem 1 and 2 can easily be adapted for (Γ_1, Γ_2) -non-zero cycles which are also long, by applying Theorem 22 with $t = 2\ell k$ instead of $t = 6k$.

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